SOMIGLIANA COORDINATES
AN ELASTICITY-DERIVED APPROACH FOR CAGE DEFORMATION

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• Sculpting brushes using fundamental solutions of elasticity
  - based on regularized Kelvinlets

Multiscale regularization [de Goes and James 2017]

Sharp fall-off [de Goes and James 2019]

Anisotropic and multi-frequency regularization [Chen and Desbrun 2022]
REAL-TIME MESH DEFORMATION

- Sculpting brushes using fundamental solutions of elasticity
  - based on regularized Kelvinlets
  - Meshfree, control of volume change, and extremely fast

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REAL-TIME MESH DEFORMATION

- Sculpting brushes using fundamental solutions of elasticity
  - based on regularized Kelvinlets
  - Meshfree, control of volume change, and extremely fast
  - Unaware of boundaries!

Multiscale regularization [de Goes and James 2017]

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REAL-TIME MESH DEFORMATION

- Cage deformer
  - based on generalized barycentric coordinates
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\[ x = \sum_{i} \phi_i(x)v_i \]
REAL-TIME MESH DEFORMATION

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x = \sum_i \phi_i(x) v_i
\]

\[
\tilde{x}(x) = \sum_i \phi_i(x) \tilde{v}_i
\]
REAL-TIME MESH DEFORMATION

Cage deformer
- based on generalized barycentric coordinates
  - many options available now

\[ x = \sum_i \phi_i(x)v_i \]
\[ \tilde{x}(x) = \sum_i \phi_i(x)\tilde{v}_i \]

Mean-value coords [Floater 2003; Ju et al. 2005; Thiery et al. 2018]
Harmonic coords [Joshi et al. 2007]
Maximum entropy coords [Hormann and Sukumar 2008]
Complex coords [Weber et al. 2009]
REAL-TIME MESH DEFORMATION

• Cage deformer
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  - Boundary-aware, meshfree, extremely fast

\[ x = \sum_i \phi_i(x)v_i \]
\[ \bar{x}(x) = \sum_i \phi_i(x)\bar{v}_i \]
REAL-TIME MESH DEFORMATION

- Cage deformer
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  - Purely geometric, no elastic feel or volume control

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    • many options available now
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\[ x = \sum_i \phi_i(x)v_i + \sum_k \psi_k(x)n_k \]

\[ \tilde{x}(x) = \sum_i \phi_i(x)\tilde{v}_i + \sum_k \psi_k(x)(c_k\tilde{n}_k) \]

Mean-value coords [Floater 2003; Ju et al. 2005; Thiery et al. 2018]

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Complex coords [Weber et al. 2009]

+ Green coordinates [Lipman et al. 2008]
CONTRIBUTIONS

• Inject *elasticity* into cage deformers for fast volumetric deformation
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  - Invariant under similarity transformations through *corotational* formulation
CONTRIBUTIONS

• Inject *elasticity* into cage deformers for fast volumetric deformation
  − *Matrix-valued coordinates*, extending Green coordinates
  − Derived from *linear elasticity* and mimicking elastic behaviors
  − Invariant under similarity transformations through *corotational* formulation
  − Control over *volume change* and *local bulge*
Green coordinates (GC)
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PDE:

$$\Delta u = 0$$
Green coordinates (GC)

PDE:
\[ \Delta u = 0 \]

Fundamental solutions:
\[ G(y, x) = \begin{cases} 
-\frac{1}{4\pi r}, & d = 3, \\
\frac{1}{2\pi} \log(r), & d = 2.
\end{cases} \]
Green coordinates (GC)

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Boundary reformulation:
\[ u(x) = \int_{\partial \Omega} \left[ \nabla_n G(y, x)u(y) - G(y, x)\nabla_n u(y) \right] d\sigma_y \]
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u(x) = x
\]

\[
\{ \phi_i(x), \psi_k(x) \} \in \mathbb{R}
\]
FROM GREEN TO SOMIGLIANA

Green coordinates (GC)

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\[ u(x) = \int_{\partial \Omega} \nabla_n G(y, x) [u(y) - G(y, x) \nabla_n u(y)] \, d\sigma_y \]

Somigliana coordinates (SC)

\[ \Delta u + \frac{1}{1 - 2\nu} \nabla (\nabla \cdot u) = 0 \]

\[ u(x) = x \]

\[ \{\phi_i(x), \psi_k(x)\} \in \mathbb{R} \]
FROM GREEN TO SOMIGLIANA

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**Somigliana coordinates (SC)**

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\[ \Delta u + \frac{1}{1 - 2\nu} \nabla (\nabla \cdot u) = 0 \]

**Fundamental solutions:**

\[ K(x, y) = \begin{cases} \frac{a-b}{r} I + \frac{b}{r^3} rr^t, & d = 3, \\ (b-a) \log(r) I + \frac{b}{r^2} rr^t, & d = 2. \end{cases} \]
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\[ u(x) = \int_{\partial\Omega} [T(y, x) u(y) + \mathcal{K}(y, x) \tau(y)] \, d\sigma_y \]
**FROM GREEN TO SOMIGLIANA**

### Green coordinates (GC)

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\[
\Delta u = 0
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**Fundamental solutions:**
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u(x) = \int_{\partial \Omega} \nabla G(y, x) u(y) - G(y, x) \nabla u(y) \, d\sigma_y
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u(x) = x
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\{\phi_i(x), \psi_k(x)\} \in \mathbb{R}
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**Boundary reformulation:**
\[
u(x) = \int_{\partial \Omega} \left[ T(y, x) u(y) + K(y, x) r(y) \right] \, d\sigma_y
\]

**Lord Kelvin**

**Carlo Somigliana**
FROM GREEN TO SOMIGLIANA

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**Boundary reformulation:**
\[ u(x) = \int_{\partial\Omega} [T(y, x) u(y) + K(y, x) r(y)] \, d\sigma_y \]

\[ \{T_i(x), K_k(x)\} \in \mathbb{R}^{d \times d} \]
Compute SC w.r.t. a triangulated cage

\[
\begin{align*}
T_i(x) &= \int_{\partial \Omega} T(y, x) \phi_i(y) d\sigma_y, \\
K_k(x) &= \int_{\partial \Omega} K(y, x) \psi_k(y) d\sigma_y.
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\]

\(T_i\) and \(K_k\) as matrix functions of Poisson ratio \(\nu\)

\[
x = \sum_i T_i(x) v_i + \sum_k K_k(x)(c n_k)
\]
Compute SC w.r.t. a triangulated cage

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\[\phi_i(y)\]
\[\psi_k(y)\]

\[T_i\]
\[K_k\]

\[v_i\]
\[n_k\]

\[x = \sum_i T_i(x) v_i + \sum_k K_k(x) (c n_k)\]

\[v=0.0\]

\[K_k(x)n_k\]
\[K_k(x)n_k^\perp\]
\[T_i(x)n_i\]
\[T_i(x)n_i^\perp\]
Compute SC w.r.t. a triangulated cage

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\[x = \sum_i T_i(x) v_i + \sum_k K_k(x)(c n_k)\]

\[T_i \text{ and } K_k \text{ as matrix functions of Poisson ratio } \nu\]
$T_i(x)$ and $K_k(x)$ are not rotationally invariant
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- Typical remedy: corotational formulation

$$\tilde{x}(x) = \left( \sum_i R_i T_i(x) R_i^t \right)^{-1} \left[ \sum_i R_i T_i(x) R_i^t \bar{v}_i + \sum_k R_k K_k(x) R_k^t \bar{\tau}_k \right]$$
$T_i(x)$ and $K_k(x)$ are **not** rotationally invariant.

- Typical remedy: corotational formulation

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$$\tilde{\tau}_k = s_k R_k n_k$$

$$= \left[ \frac{2(1 - \nu)}{1 - 2\nu} \eta_k + \frac{2\nu(d - 1)}{1 - 2\nu} \lambda_k \right] R_k n_k$$
\( T_i(x) \) and \( K_k(x) \) are not rotationally invariant

- Typical remedy: corotational formulation

\[
\tilde{x}(x) = \left( \sum_i R_i T_i(x) R_i^T \right)^{-1} \left[ \sum_i R_i T_i(x) R_i^T \tilde{v}_i + \sum_k R_k K_k(x) R_k^T \tilde{\tau}_k \right]
\]

\[
\tilde{\tau}_k = s_k R_k n_k
\]

\[
= \left[ \frac{2(1-\nu)}{1-2\nu} \eta_k + \frac{2\nu(d-1)}{1-2\nu} \lambda_k \right] R_k n_k
\]

Estimate \( \{R_k, \lambda_k, \eta_k\} \) for each boundary facet
GLOBAL VS. LOCAL ROTATION & TANGENT STRETCHES

Rest pose

Global variant

\[ R_k = R_{\text{global}}, \lambda_k = \lambda_{\text{global}} \text{ from the optimal similarity transformation} \]
GLOBAL VS. LOCAL ROTATION & TANGENT STRETCHES

Rest pose

Global variant

Local variant

$R_k = R_{\text{global}}, \lambda_k = \lambda_{\text{global}}$ from the optimal similarity transformation

$R_k$ and $\lambda_k$ are decided on per facet basis
GLOBAL VS. LOCAL ROTATION & TANGENT STRETCHES

Rest pose

Global variant

In between

Local variant

$R_k = R_{global}, \lambda_k = \lambda_{global}$ from the optimal similarity transformation

blend global and local $R_k, \lambda_k$

$R_k$ and $\lambda_k$ are decided on per facet basis
CURVATURE-BASED NORMAL STRETCHES

- Normal stretching factor $\eta_k$ for each cage facet
  - No information about out-of-plane deformation
  - E.g., account for curvature change for local bulging
    $$\eta_k = \lambda_k \exp(\gamma \beta_k / (2^{d-1} \pi))$$
  - Compute on-the-fly

Small $\gamma$  
Large $\gamma$
CURVATURE-BASED NORMAL STRETCHES

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  - No information about out-of-plane deformation
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![Diagram showing normal stretching factor calculation and deformation examples]
CURVATURE-BASED NORMAL STRETCHES

- Normal stretching factor $\eta_k$ for each cage facet
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- Our choice of $R_j, \lambda_j, \eta_j$ keeps the deformation invariant under similarity transformations
  $$\mathbf{x}(sR\mathbf{x} + t) = sR\mathbf{x}(\mathbf{x}) + t$$
• SC is equivalent to GC in 2D, for $\nu = \infty$, and $\gamma = 0$
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2D SC deformation with $\nu = \infty$, $\gamma = 0$

\[
\bar{x}(x) = \frac{1}{2\pi} \sum_{e} \int_{0}^{L_e} \left( \frac{r t}{r^2} I + \frac{1}{r^2} (nr' - rn') \right) \tilde{y} \, d\sigma_y
\]
\[
= \frac{1}{2\pi} \sum_{e} \int_{0}^{L_e} \frac{1}{r^2} \begin{pmatrix} r_1 n_1 + r_2 n_2 & 0 \\ 0 & r_1 n_1 + r_2 n_2 \end{pmatrix} \tilde{y} \, d\sigma_y
\]
\[
+ \begin{pmatrix} 0 & r_2 n_1 - r_1 n_2 \\ r_1 n_2 - r_2 n_1 & 0 \end{pmatrix} \tilde{y} \, d\sigma_y
\]

Expressed in complex numbers

\[
\frac{1}{2\pi} \int_{0}^{L_e} \frac{r_1 n_1 + r_2 n_2 + i(r_1 n_2 - r_2 n_1)}{r^*r} \tilde{y} \, d\sigma_y
\]
\[
= \frac{1}{2\pi} \int_{0}^{L_e} \frac{(r_1 - i r_2)(n_1 + i n_2)}{r^*r} \tilde{y} \, d\sigma_y
\]
\[
= \frac{1}{2\pi} \int_{0}^{L_e} \frac{r^* r \tilde{y}}{r^*r} \tilde{y} \, d\sigma_y = \frac{1}{2\pi} \int_{0}^{L_e} \frac{i \cdot \tilde{y}}{r} \, d\sigma_y
\]
\[
= \frac{1}{2\pi} \int_{0}^{L_e} \frac{\tilde{y}}{r} \, d\sigma_y
\]

Through Cauchy integral formula

\[
g_{s,j}(x) = \sum_{j=1}^{i} C_j(z) f_j
\]
\[
C_j(z) = \frac{1}{2\pi} \left( \frac{B_{s,j}(z)}{A_{j,s}} \log \left( \frac{B_{s,j}(z)}{B_{j,s}(z)} \right) - \frac{B_{j,s}(z)}{A_{j,s}} \log \left( \frac{B_{j,s}(z)}{B_{s,j}(z)} \right) \right)
\]

Cauchy-Green complex barycentric coordinates

Theorem 3: Lipman's 2D Green coordinates [LLCO08] are identical to discrete Cauchy coordinates.

[Weber et al. 2009]
\[ K_k(x) = \int_{\Delta_k} K(y, x) \, d\sigma_y = 2|\Delta_k| \int_0^1 \, d\alpha \int_0^{1-\alpha} \, d\beta \, K(y(\alpha, \beta), x) \]
\[
K_k(x) = \int_{\Delta_k} \mathcal{K}(y, x) \, d\sigma_y = 2|\Delta_k| \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, \mathcal{K}(y(\alpha, \beta), x)
\]

\[
2|\Delta_k| \sum_j w_j \left( \frac{a-b}{r(\alpha_j, \beta_j)} I + \frac{b}{r^3(\alpha_j, \beta_j)} r(\alpha_j, \beta_j) r^t(\alpha_j, \beta_j) \right)
\]
IMPLEMENTATION

\[
K_k(x) = \int_{\Delta_k} K(y, x) \, d\sigma_y = 2|\Delta_k| \int_0^1 \alpha \int_0^{1-\alpha} \beta K(y(\alpha, \beta), x)
\]

- #query points: 39k
- #quadratures per face: 7500
- #cage faces: 184

\[2|\Delta_k| \sum_j w_j \left( \frac{a - b}{r(\alpha_j, \beta_j)} \mathbf{I} + \frac{b}{r^3(\alpha_j, \beta_j)} \mathbf{r}(\alpha_j, \beta_j)\mathbf{r}^t(\alpha_j, \beta_j) \right)\]

\(\approx\) Coord. computation time: 1.5s
RESULTS
VOLUME PRESERVING VS. LOCAL BULGING

- Better volume preservation ($\nu$)
- Larger local bulging ($\gamma$)

Input GC

$v=0, \gamma=0$
$v=0.3, \gamma=0$
$v=0.48, \gamma=0$

$v=0, \gamma=17$
$v=0.3, \gamma=12$
$v=0.48, \gamma=3$
2D COMPARISONS

input

MVC

GC

Ours (global)
2D COMPARISONS
3D COMPARISONS

input

MVC

GC

Ours
3D COMPARISONS
COMPARISONS

BEM

Ours (Global variant)

BEM

Ours (Global variant)

input
FUTURE WORKS

- Cage triangulation may break the symmetry
  - Compute SC on quad meshes
  - Adopt other bulging factors less sensitive to the triangulation
FUTURE WORKS

• Cage triangulation may break the symmetry
  – Compute SC on quad meshes
  – Adopt other bulging factors less sensitive to the triangulation

• Accelerate SC computations
  – Adaptive quadrature rules for far- and near-field evaluations
  – Derive closed-form expressions
FUTURE WORKS

• Cage triangulation may break the symmetry
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  − Adaptive quadrature rules for far- and near-field evaluations
  − Derive closed-form expressions

• Space-time cages for real-time animation editing