Somigliana Coordinates: an elasticity-derived approach for cage deformation

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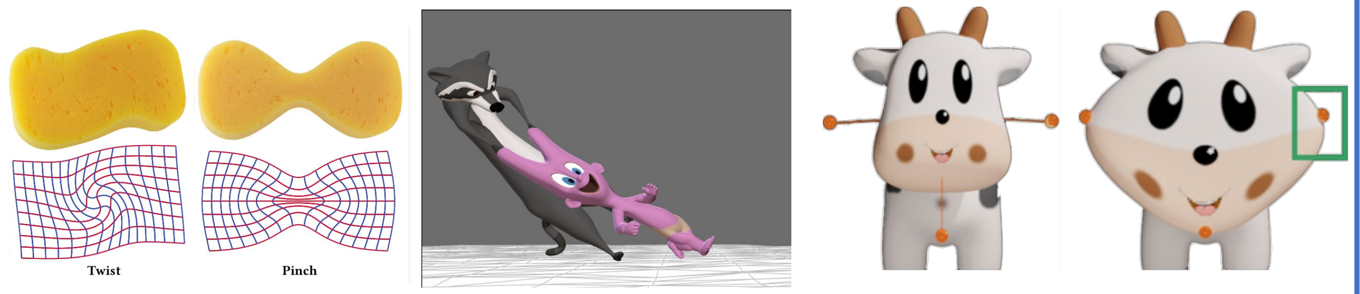
Abstract

In this paper, we present a novel cage deformer based on elasticity-derived matrix-valued coordinates. In order to bypass the typical shearing artifacts and lack of volume control of existing cage deformers, we promote a more elastic behavior of the cage deformation by deriving our coordinates from the Somigliana identity, a boundary integral formulation based on the fundamental solution of linear elasticity. Given an initial cage and its deformed pose, the deformation of the cage interior is deduced from these Somigliana coordinates via a corotational scheme, resulting in a matrix-weighted combination of both vertex positions and face normals of the cage. Our deformer thus generalizes Green coordinates, while producing physically-plausible spatial deformations that are invariant under similarity transformations and with interactive bulging control. We demonstrate the efficiency and versatility of our method through a series of examples in 2D and 3D.

Background

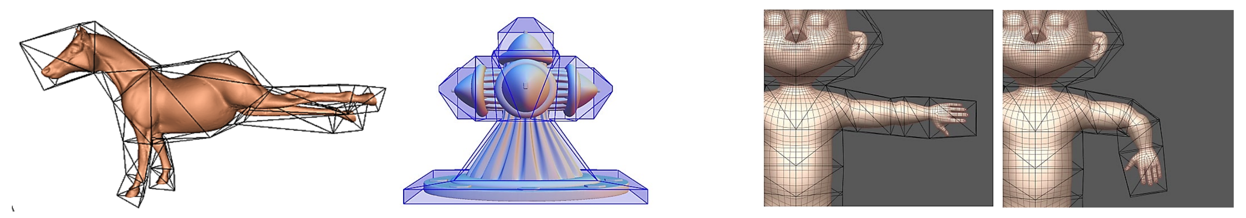
Real-time deformation techniques:

- Sculpting brushes via **fundamental solutions** of elasticity (**Kelvinlets**)

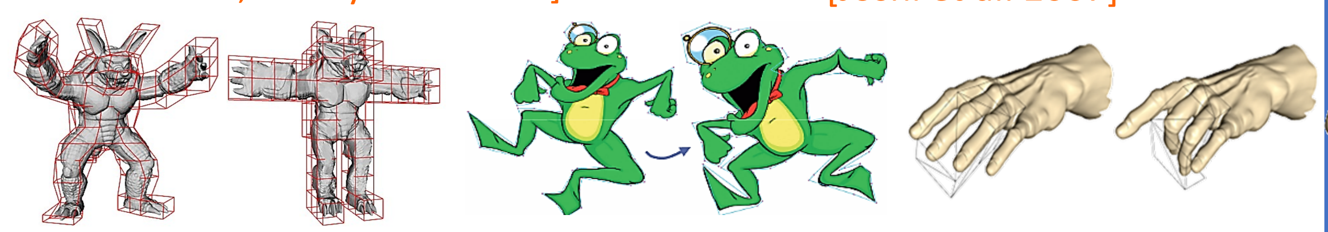


[de Goes and James 2017, 2019] [Chen and Desbrun 2022]

- Cage deformers based on **generalized barycentric coordinates**



Mean-value coord. [Floater 2003; Ju et al. 2005; Thiery et al. 2018] **Harmonic coord.** [Joshi et al. 2007]



Maximum entropy coord. [Hormann and Sukumar 2008] **Complex coord.** [Weber et al. 2009] **Green coord.** [Lipman et al. 2008]

Somigliana Coordinates

Our Somigliana coordinates has a similar derivation as Green coordinates based on the Somiglina identity for linear elasticity, an analogue to the Green's third identity for the harmonic equation.

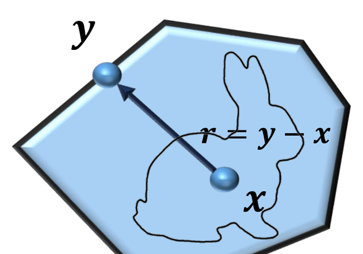
Green coordinates (GC)

PDE: $\Delta \boldsymbol{u} = \boldsymbol{0}$

Fundamental solutions:
$$G(\boldsymbol{y},\boldsymbol{x}) = \begin{cases} -\dfrac{1}{4\pi r}, & d=3,\\ \dfrac{1}{2\pi}\log(r), & d=2.\end{cases}$$

Boundary reformulation:
$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\partial\Omega} [\nabla_{\boldsymbol{n}} G(\boldsymbol{y},\boldsymbol{x})\boldsymbol{u}(\boldsymbol{y}) - G(\boldsymbol{y},\boldsymbol{x})\nabla_{\boldsymbol{n}}\boldsymbol{u}(\boldsymbol{y})]\, d\sigma_{\boldsymbol{y}}$$

$$\nabla_n G(\boldsymbol{y},\boldsymbol{x}) = \frac{1}{2^{d-1}\pi}\frac{\boldsymbol{r}^t\boldsymbol{n}}{r^d}$$



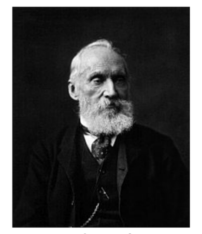
Somigliana coordinates (SC)

$$\Delta \boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{0}$$

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \dfrac{a-b}{r}\boldsymbol{I} + \dfrac{b}{r^3}\boldsymbol{r}\boldsymbol{r}^t, & d=3,\\ (b-a)\log(r)\boldsymbol{I} + \dfrac{b}{r^2}\boldsymbol{r}\boldsymbol{r}^t, & d=2.\end{cases}$$

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\partial\Omega} [\mathcal{T}(\boldsymbol{y},\boldsymbol{x})\boldsymbol{u}(\boldsymbol{y}) + \mathcal{K}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\tau}(\boldsymbol{y})]\, d\sigma_{\boldsymbol{y}}$$

$$\mathcal{T}(\boldsymbol{y},\boldsymbol{x}) = \frac{\mu(a-2b)}{r^d}[(\boldsymbol{n}^t\boldsymbol{r})\boldsymbol{I} + \boldsymbol{n}\boldsymbol{r}^t - \boldsymbol{r}\boldsymbol{n}^t] + \frac{2\mu b d}{r^{d+2}}(\boldsymbol{n}^t\boldsymbol{r})\boldsymbol{r}\boldsymbol{r}^t$$



Lord Kelvin

Carlo Somigliana

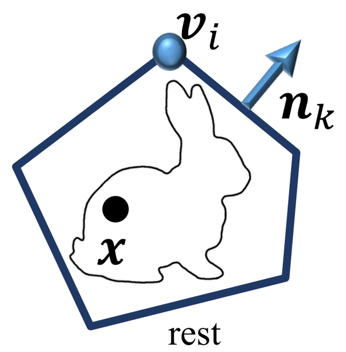
Identity mapping

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$$

$$\{\phi_i(\boldsymbol{x}), \psi_k(\boldsymbol{x})\} \in \mathbf{R} \qquad \{T_i(\boldsymbol{x}), K_k(\boldsymbol{x})\} \in \mathbf{R}^{d\times d}$$

Corotational formulation

The corotational formulation is derived to ensure the cage deformation is invariant to similarity transformations.



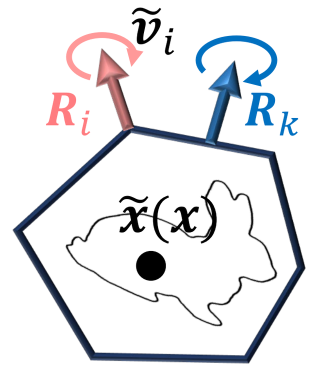
At the rest pose
$$\sum_i T_i(\boldsymbol{x})(\boldsymbol{v}_i - \boldsymbol{x}) + \sum_k K_k(\boldsymbol{x})(c\boldsymbol{n}_k) = \boldsymbol{0}$$

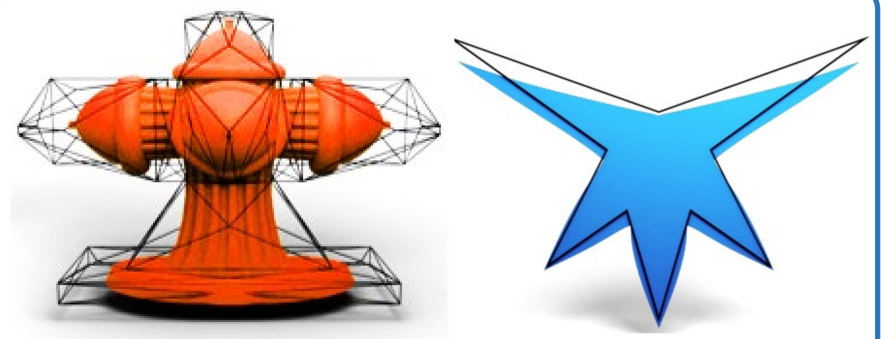
$$\tilde{\boldsymbol{x}}(\boldsymbol{x}) = \left(\sum_i R_i T_i(\boldsymbol{x}) R_i^t\right)^{-1}\left[\sum_i R_i T_i(\boldsymbol{x}) R_i^t \tilde{\boldsymbol{v}}_i + \sum_k R_k K_k(\boldsymbol{x}) R_k^t \tilde{\boldsymbol{\tau}}_k\right]$$

$$\tilde{\boldsymbol{\tau}}_k = \tilde{\boldsymbol{\sigma}}_k\cdot\boldsymbol{n}_k = 2R_k\left[S_k + \left(\frac{\nu}{1-2\nu}\right)\mathrm{tr}(S_k)\boldsymbol{I}\right]\boldsymbol{n}_k$$

$$S_k = \eta_k\boldsymbol{n}_k\boldsymbol{n}_k^t + \lambda_k(\boldsymbol{I} - \boldsymbol{n}_k\boldsymbol{n}_k^t)$$

$$\tilde{\boldsymbol{\tau}}_k = 2\left[\eta_k + \frac{\nu}{1-2\nu}(\eta_k + (d-1)\lambda_k)\right] R_k\boldsymbol{n}_k = s_k R_k \boldsymbol{n}_k$$

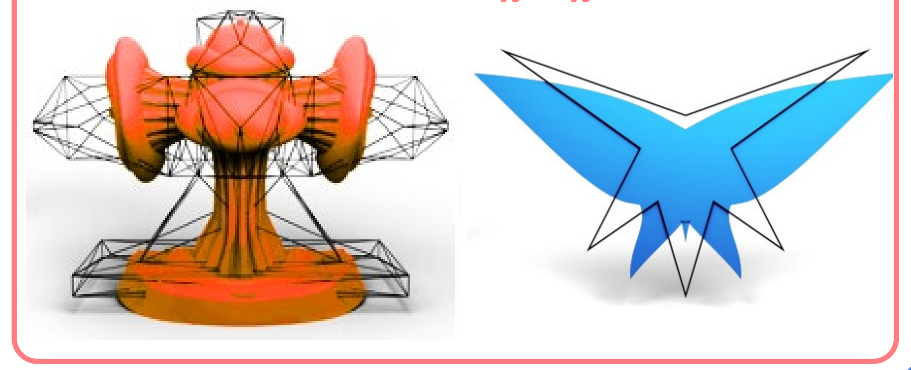




Global R_k, s_k

vs.

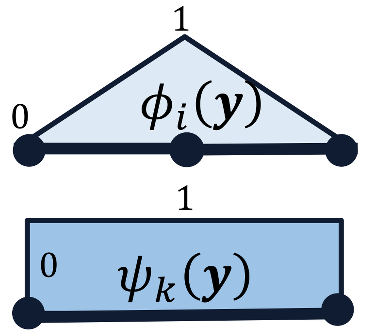
Local R_k, s_k

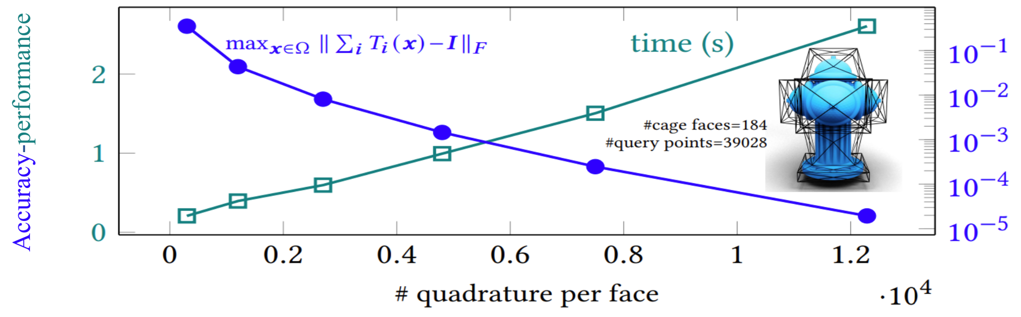


Implementations

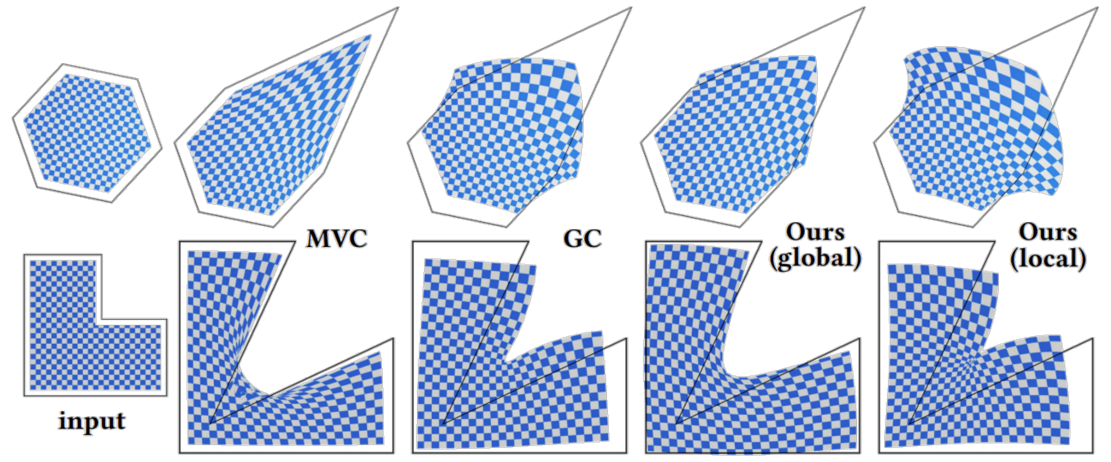
We apply quadratures to compute SC over a triangulated cage.

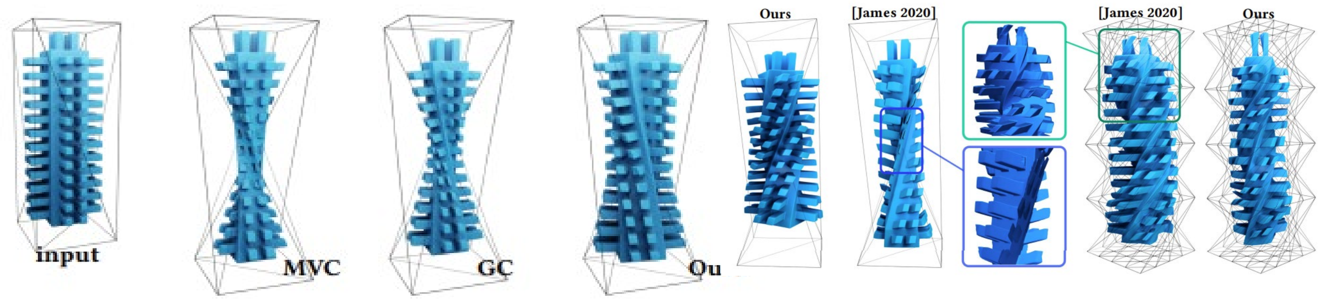
$$\begin{cases} T_i(\boldsymbol{x}) = \int_{\partial\Omega} \mathcal{T}(\boldsymbol{y},\boldsymbol{x})\phi_i(\boldsymbol{y})\,d\sigma_{\boldsymbol{y}},\\ K_k(\boldsymbol{x}) = \int_{\partial\Omega} \mathcal{K}(\boldsymbol{y},\boldsymbol{x})\psi_k(\boldsymbol{y})\,d\sigma_{\boldsymbol{y}}.\end{cases}$$





Comparisons





Results