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Material-adapted Refinable Basis Functions for Elasticity Simulation

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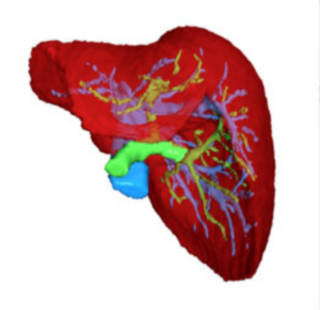
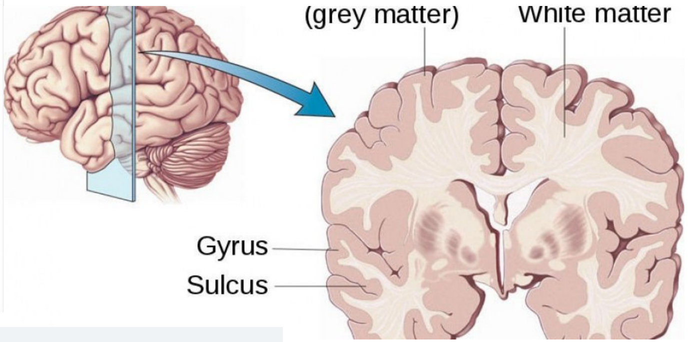
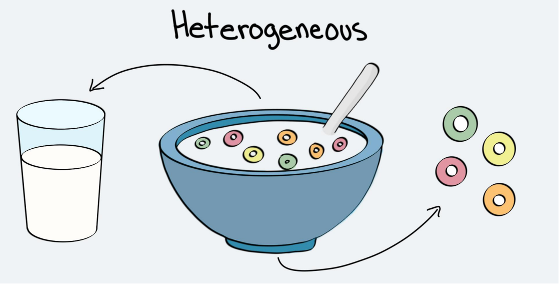
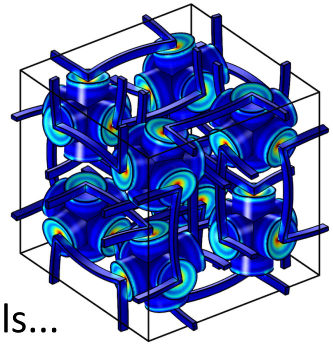
Motivation

Inhomogeneity everywhere…







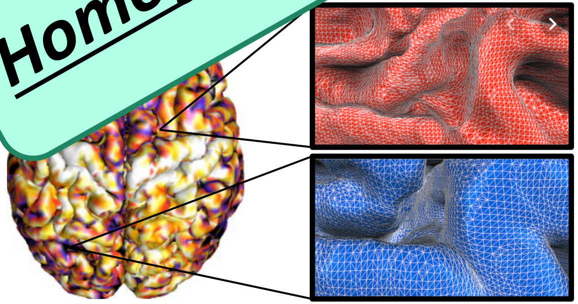


Organs, compounds, metamaterials…

Simulating inhomogeneous material objects ca very challenging

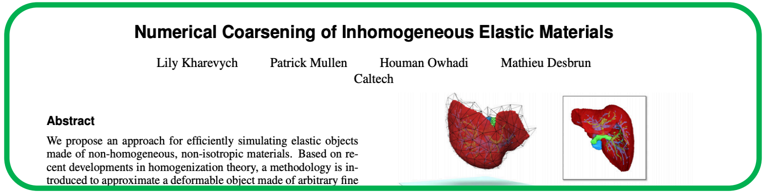
- Large
- D

Homogenization!



Homogenization (Coarse-graining)

- Fitting locally-homogeneous model

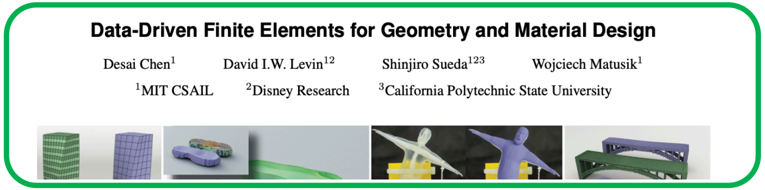


Numerical Coarsening of Inhomogeneous Elastic Materials

Lily Kharevych Patrick Mullen Houman Owhadi Mathieu Desbrun
Caltech

Abstract

We propose an approach for efficiently simulating elastic objects made of non-homogeneous, non-isotropic materials. Based on recent developments in homogenization theory, a methodology is introduced to approximate a deformable object made of arbitrary fine



Data-Driven Finite Elements for Geometry and Material Design

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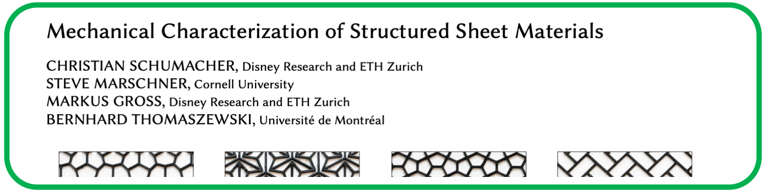
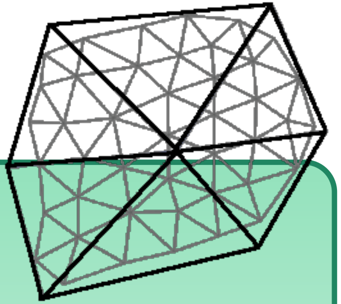
[1]MIT CSAIL [2]Disney Research [3]California Polytechnic State University



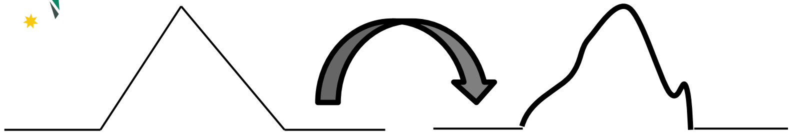
Mechanical Characterization of Structured Sheet Materials

CHRISTIAN SCHUMACHER, Disney Research and ETH Zurich
STEVE MARSCHNER, Cornell University
MARKUS GROSS, Disney Research and ETH Zurich
BERNHARD THOMASZEWSKI, Université de Montréal



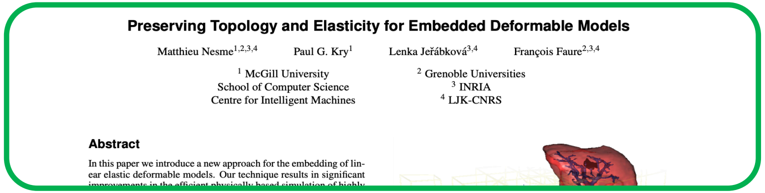
- Idea
  - “Average” the inhomogeneous potential functional **from fine to coarse**
- Limitations
  - Hard to encode general anisotropy for nonlinear problems
  - Limited ability to capture complex anisotropic behavior

Homogenization (Coarse-graining)



- Idea
  - Optimize **coarse-to-fine** ”prediction” by adapting the bases to the inhomogeneity
- Limitations
  - Limited expressivity for elasticity
  - Not flexible enough to handle arbitrary coarse scale
  - Cost much to compute

- Construct local basis functions



Preserving Topology and Elasticity for Embedded Deformable Models

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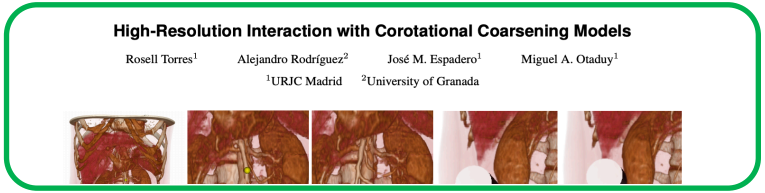
Abstract

In this paper we introduce a new approach for the embedding of linear elastic deformable models. Our technique results in significant



High-Resolution Interaction with Corotational Coarsening Models

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Numerical Coarsening using Discontinuous Shape Functions

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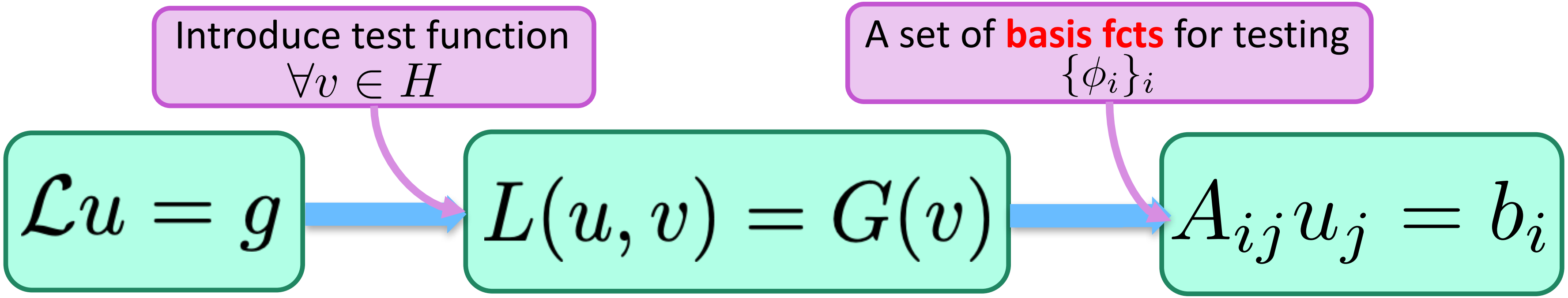
JIN HUANG*, State Key Lab of CAD&CG, Zhejiang University

How basis functions play a role



Introduce test function $\forall v \in H$

A set of **basis fcts** for testing $\{\phi_i\}_i$

$$\mathcal{L}u = g$$

$$L(u, v) = G(v)$$

$$A_{ij}u_j = b_i$$

PDE

$$\mathcal{L} : H \to H^*$$

linear

self-adjoint

positive-definite

Weak form

$$L(u, v) := \int u \cdot \mathcal{L}v$$

Lax-Milgram theorem ensures the unique solution for any RHS

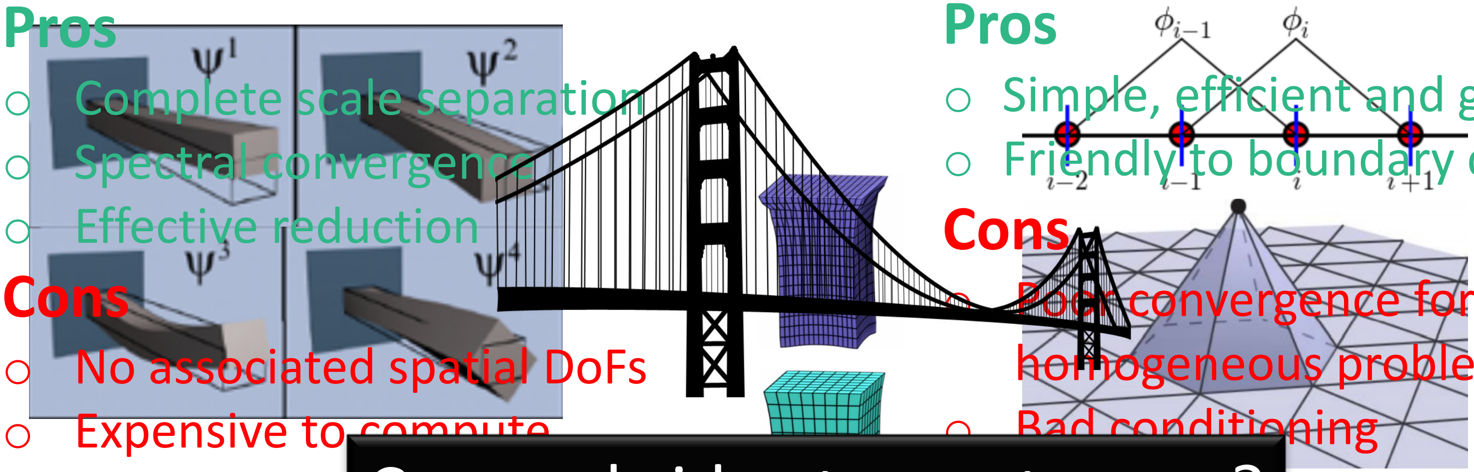
Linear Eqns

$$A_{ij} = L(\phi_i, \phi_j)$$

$$b_i = G(\phi_i)$$

Choosing basis functions

- Global basis functions
  - ➢ Eigenfunctions (modal basis)
- Local basis functions
  - ➢ Low-order polynomial basis



Pros

- ○ Complete scale separation
- ○ Spectral convergence
- ○ Effective reduction

Cons

- ○ No associated spatial DoFs
- ○ Expensive to compute

Pros

- ○ Simple, efficient and generalizable
- ○ Friendly to boundary conditions

Cons

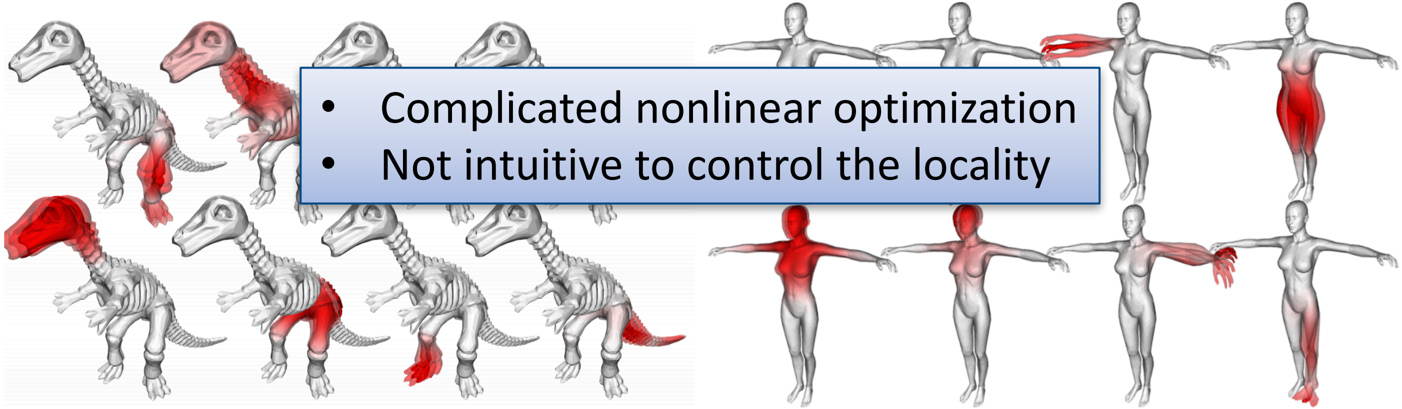
- ○ Poor convergence for even homogeneous problem
- ○ Bad conditioning

Can we bridge two extremes?

Sparse optimization approach

- [Brandt and Hildebrandt 2017]

$$\mathbf{u}_i := \begin{cases} \arg\min_{\mathbf{u}} \ \mathbf{u}^T\mathbf{H}\mathbf{u} + \boxed{\mu\|\mathbf{u}\|_1} \\ \text{subject to } \ \mathbf{u}^T\mathbf{M}\mathbf{u} = 1 \text{ and } \forall j < i : \ \mathbf{u}^T\mathbf{M}\mathbf{u}_j = 0 \end{cases}$$



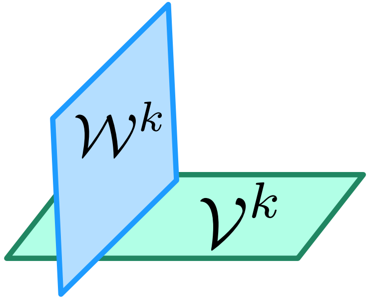
- Complicated nonlinear optimization
- Not intuitive to control the locality

MultiResolution Analysis (MRA)

- Hierarchical orthogonal decomposition

$$\mathcal{V}^{k+1} = \mathcal{V}^{k} \oplus \mathcal{W}^{k}$$

$$\boxed{\mathcal{V}^{q} = \mathcal{V}^{1} \oplus \mathcal{W}^{1} \oplus \cdots \oplus \mathcal{W}^{q-1}}$$



- Multiresolutional basis functions

$$\underbrace{\mathcal{V}^{k+1}}_{\text{span}(\varphi_i^{k+1})} = \underbrace{\mathcal{V}^{k}}_{\text{span}(\varphi_i^{k})} \oplus \underbrace{\mathcal{W}^{k}}_{\text{span}(\psi_j^{k})} \qquad \forall i, j, \int \varphi_i^k \psi_j^k = 0$$

Scaling functions

Wavelets

MultiResolution Analysis (MRA)

- Multiresolutional upsampling

$$u^q(x) = \boxed{\sum_i v_i^1 \varphi_i^1(x)} + \boxed{\sum_{k=1}^{q-1} \sum_j w_j^k \psi_j^k(x)}$$

Missing details

Coarsest solution

- Stiffness matrix structure

$$\begin{bmatrix} A^1 := L(\varphi^1, \varphi^1) & L(\varphi^1, \psi^1) & \ldots & L(\varphi^1, \psi^{q-1}) \\ L(\psi^1, \varphi^1) & B^1 := L(\psi^1, \psi^1) & \ldots & L(\psi^1, \psi^{q-1}) \\ \vdots & \vdots & \ddots & \vdots \\ L(\psi^{q-1}, \varphi^1) & L(\psi^{q-1}, \psi^1) & \ldots & B^{q-1} := L(\psi^{q-1}, \psi^{q-1}) \end{bmatrix}$$

Orthogonality

Why L_2-orthogonality?

Operator-orthogonal decomposition

$$\underbrace{\mathcal{V}^{k+1}}_{\mathrm{span}(\varphi_i^{k+1})} = \underbrace{\mathcal{V}^{k}}_{\mathrm{span}(\varphi_i^{k})} \oplus_{\mathcal{L}} \underbrace{\mathcal{W}^{k}}_{\mathrm{span}(\psi_j^{k})}$$

$$\forall i, j, \int \varphi_i^k \mathcal{L} \psi_j^k = 0$$

❯

Block diagonal stiffness matrix

$$\mathbb{L} = \begin{pmatrix} \mathbb{A}^1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbb{B}^1 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbb{B}^{q-1} \end{pmatrix}$$

Clustered scale separation

Spatial locality

Besides, we want the basis functions to be locally supported

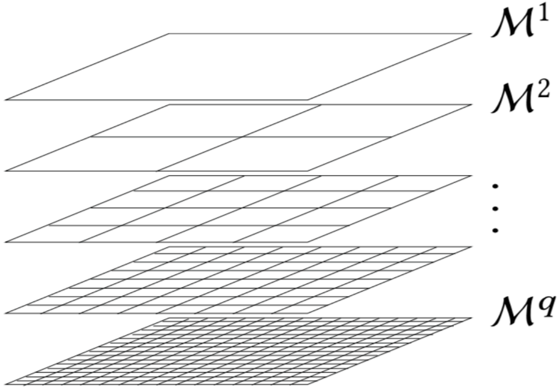
- to be able to capture local deformation
- to handle boundary conditions properly
- to sparsify the system matrix for computational efficiency

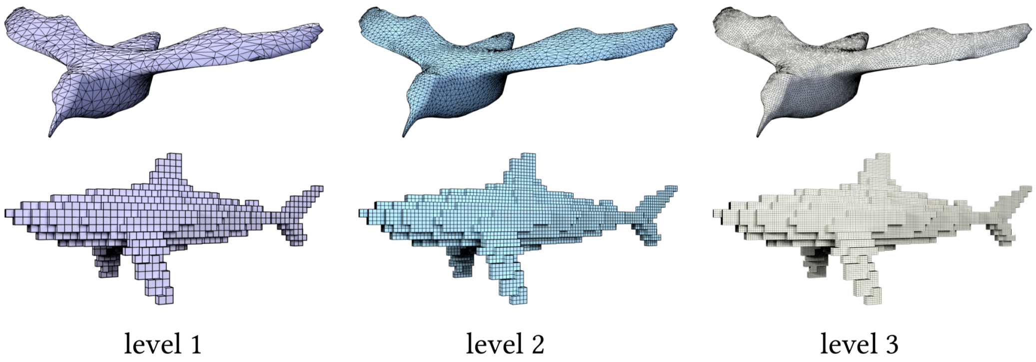
L-orthogonality + **Spatial locality** = **Space- & eigenspace-localized basis fcts**

Construction for elasticity

Mesh hierarchy

- Loose requirement on mesh hierarchy…
  - Simplicial/polyhedral
  - Nested/non-nested
  - Subdivision/aggregation

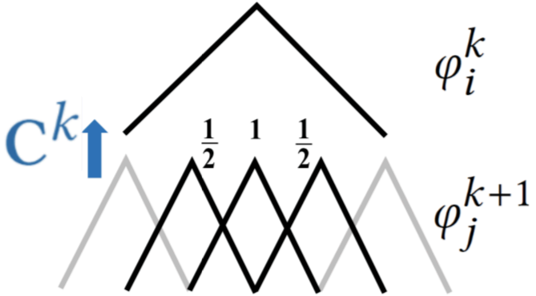




Refinable basis functions

- …as long as the associated basis functions are ***refinable***

$$\forall k \in \{1, .., q-1\}, \quad \varphi_i^k = \sum_{j=1}^{n_{k+1}} \mathbf{c}_{ij}^k \, \varphi_j^{k+1},$$



L_2 orthogonal by construction

- Refinement kernel W

$$\mathbf{c}^k \mathbf{W}^{k,T} = 0 \xrightarrow[\text{wavelet}]{\text{parameterize}} \psi_i^k = \sum_{j=1}^{n_{k+1}} \mathbf{W}_{ij}^k \varphi_j^{k+1}$$

Matrix-valued extension

- Matrix-valued basis functions [Chen 2018]

For any level k

$$\boxed{\varphi_i^k : \Omega \longrightarrow \mathbb{R}^{d\times d}}$$

Idea: provide sufficient DOFs to encode local anisotropy.

Finest level

$$\varphi_i^q(x) = \begin{bmatrix} \bar{\varphi}_i(x) & 0 & 0 \\ 0 & \bar{\varphi}_i(x) & 0 \\ 0 & 0 & \bar{\varphi}_i(x) \end{bmatrix}$$

- Matrix dimensions

$$\mathbf{C}_{ij}^k, \mathbf{W}_{ij}^k \in \mathbb{R}^{3\times 3}$$

$$\mathbf{C}^k \in \mathbb{R}^{3n_k \times 3n_{k+1}}$$

$$\mathbf{W}^k \in \mathbb{R}^{(3n_{k+1}-3n_k)\times 3n_{k+1}}$$

Material-adapted refinement

Material blind MR bases

$$\varphi_i^k = \sum_{j=1}^{n_{k+1}} \mathrm{C}_{ij}^k\, \varphi_j^{k+1}$$

$$\psi_i^k = \sum_{j=1}^{n_{k+1}} \mathbf{W}_{ij}^k \varphi_j^{k+1}$$

Bootstrapping

$$\boldsymbol{\varphi}_i^q \equiv \varphi_i^q$$

Material adapted MR bases

$${}^{k}_{i} = \sum_{j=1}^{n_{k+1}} \underbrace{\mathbb{C}_{ij}^k}_{\text{To be solved...}} \boldsymbol{\varphi}_j^{k+1},$$

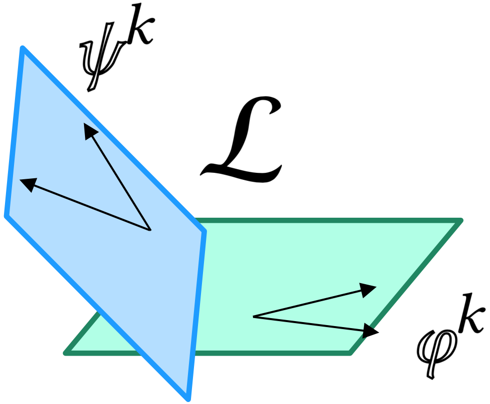
$$\boldsymbol{\psi}_i^k = \sum_{j=1}^{n_{k+1}} \mathbf{W}_{ij}^k\, \boldsymbol{\varphi}_j^{k+1}$$

L-orthogonality

Enforce orthogonality w.r.t. metric $\mathcal{L}$

$$\int_\Omega \varphi_i^k \mathcal{L} \psi_j^{k,T} = 0 \quad \forall i, j,$$

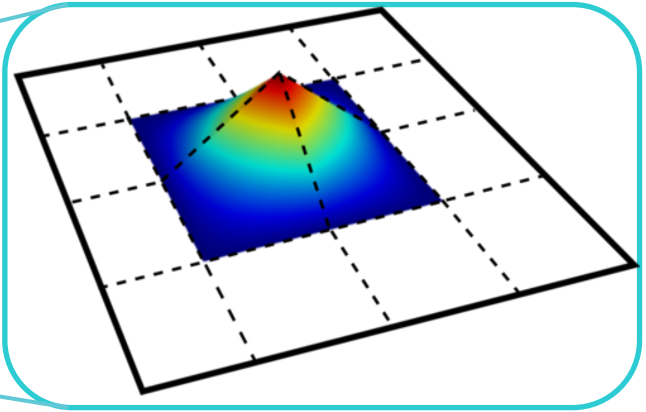
$$\boxed{\mathbb{C}^k \mathbb{A}^{k+1} W^{k,T} = \mathbf{0}}$$



Spatial locality

Enforce collocation with non-adapted local basis functions

$$\int_{\Omega} \varphi_i^k \boxed{\varphi_j^k} = \delta_{ij} \qquad \forall i, j,$$



$$\boxed{\mathbf{C}^k \mathbb{C}^{k,T} = \mathbb{I}}$$

Variational formulation

Equivalent variational formulation

$$\varphi_i = \arg\min_{\phi} \int_\Omega \phi \mathcal{L} \phi \qquad \text{s.t.} \qquad \int_\Omega \phi \varphi_j = \delta_{ij}\ \forall j.$$

[Refer to our paper for a proof]

Discrete form

$$\mathbb{C}^k = \arg\min_M \mathrm{Tr}\left[M \mathbb{A}^{k+1} M^T\right] \quad \text{s. t.} \quad M\,\mathbf{C}^{k,T} = \mathbb{I}_{3n_k \times 3n_k}.$$

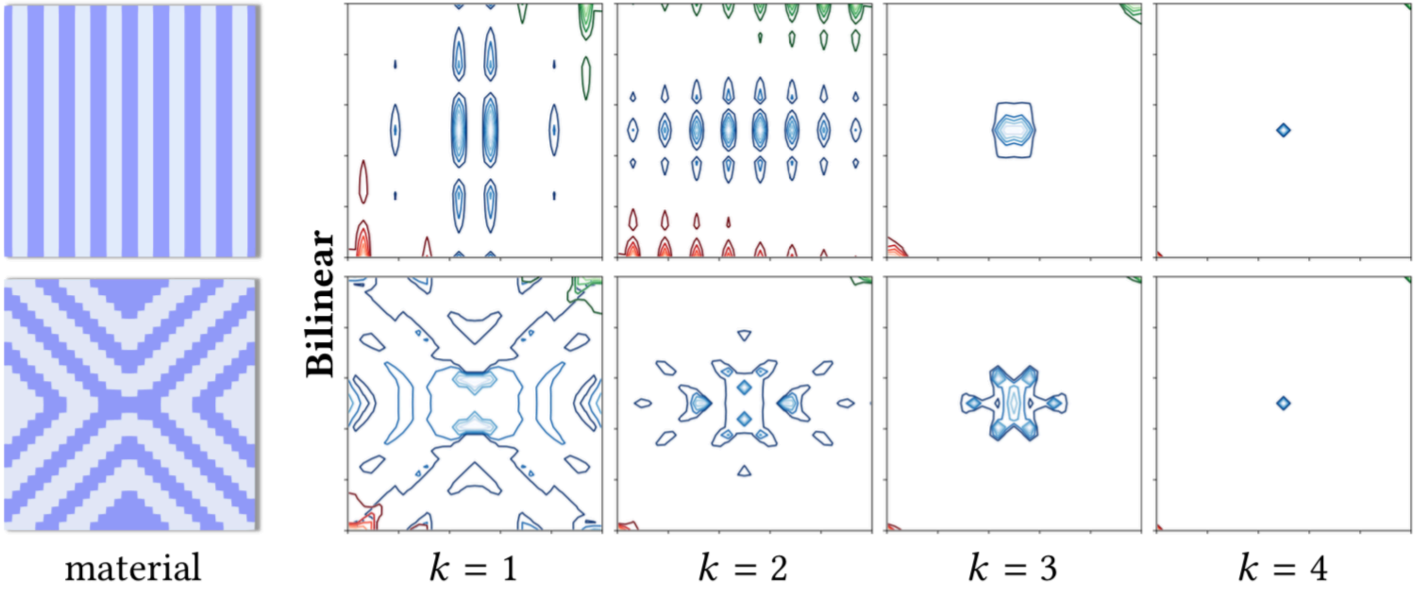
[A simple quadratic problem]

Close-formed solution

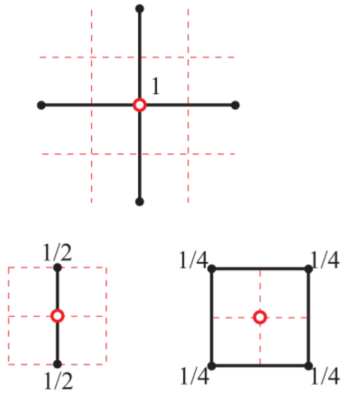
$$\mathbb{C}^k = \mathbf{C}^{k,\dagger}\left[\mathbb{I}_{3n_{k+1}\times 3n_{k+1}} - \mathbb{A}^{k+1}\,\mathbf{W}^{k,T}\left(\underbrace{\mathbb{B}^k}_{\mathbf{W}^k \mathbb{A}^{k+1} \mathbf{W}^{k,T}}\right)^{-1}\mathbf{W}^k\right],$$

[Recursively applied for multilevel decomposition]

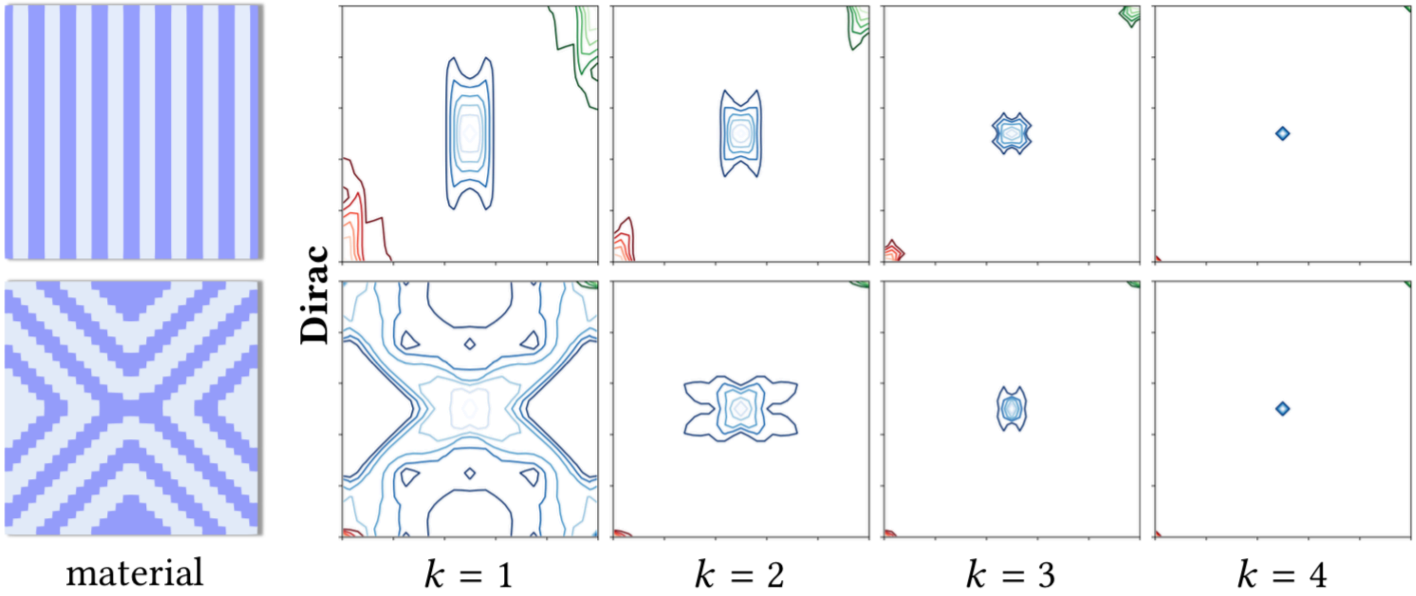
Hierarchical adapted basis fcts

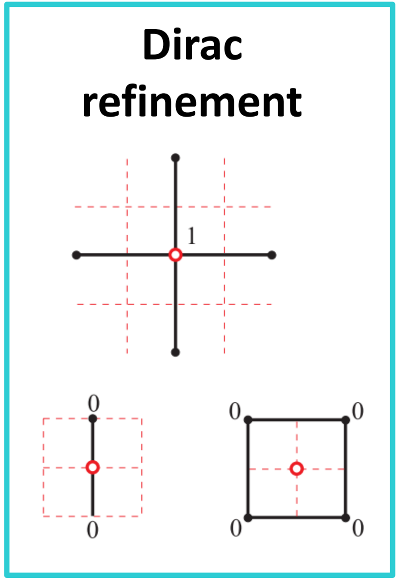


Bilinear refinement



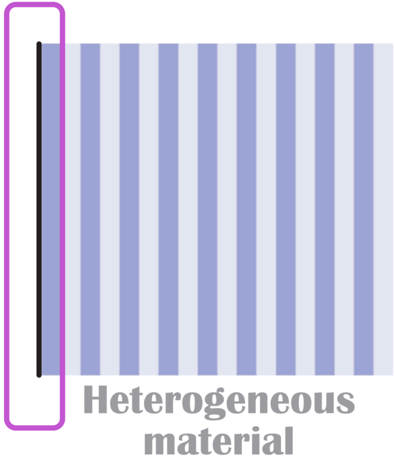
Hierarchical adapted basis fcts

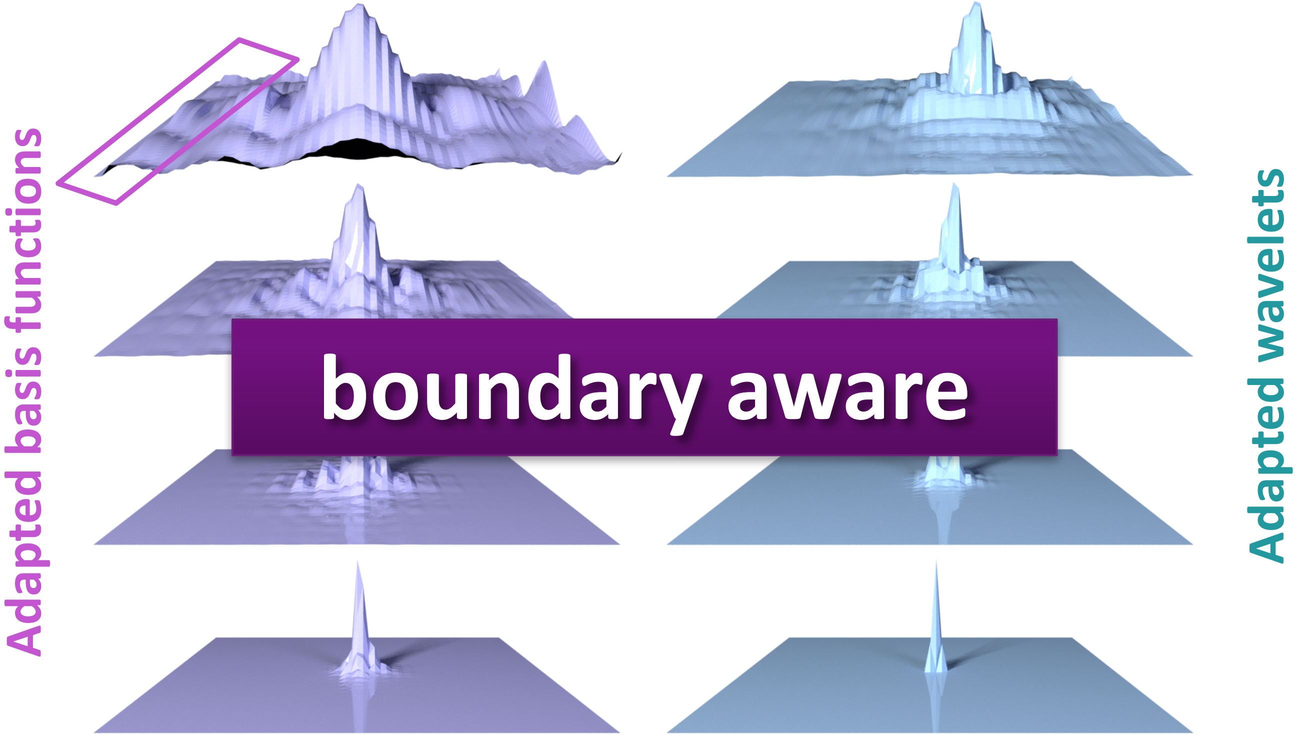




Dirac refinement

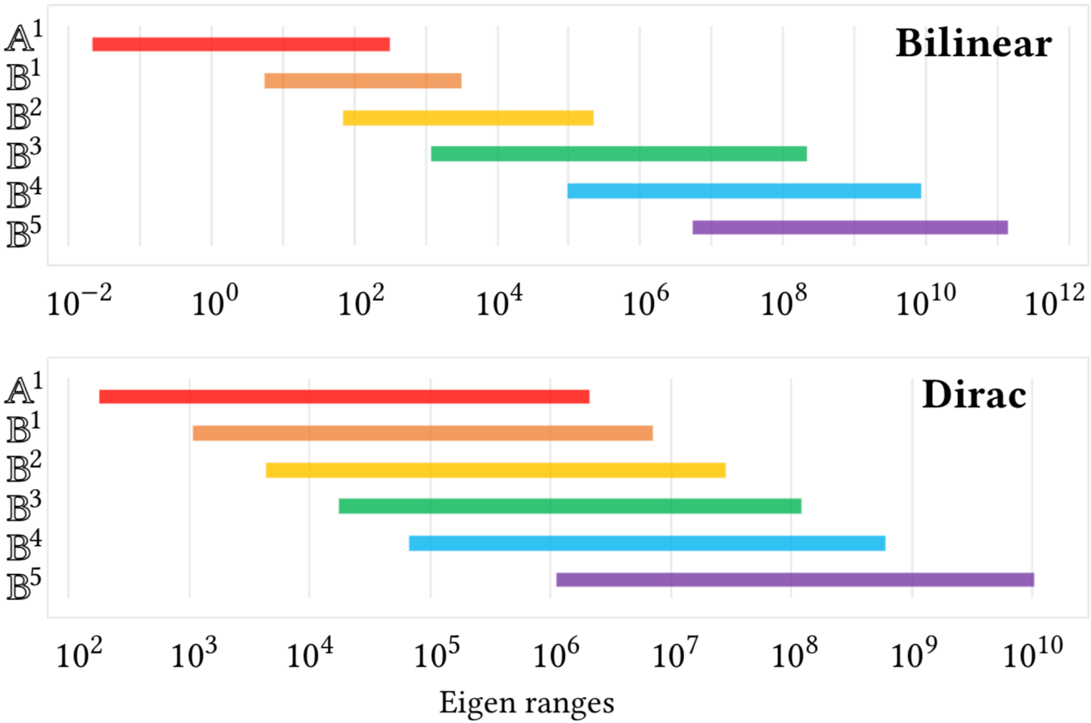
Hierarchical adapted basis fcts & wavelets

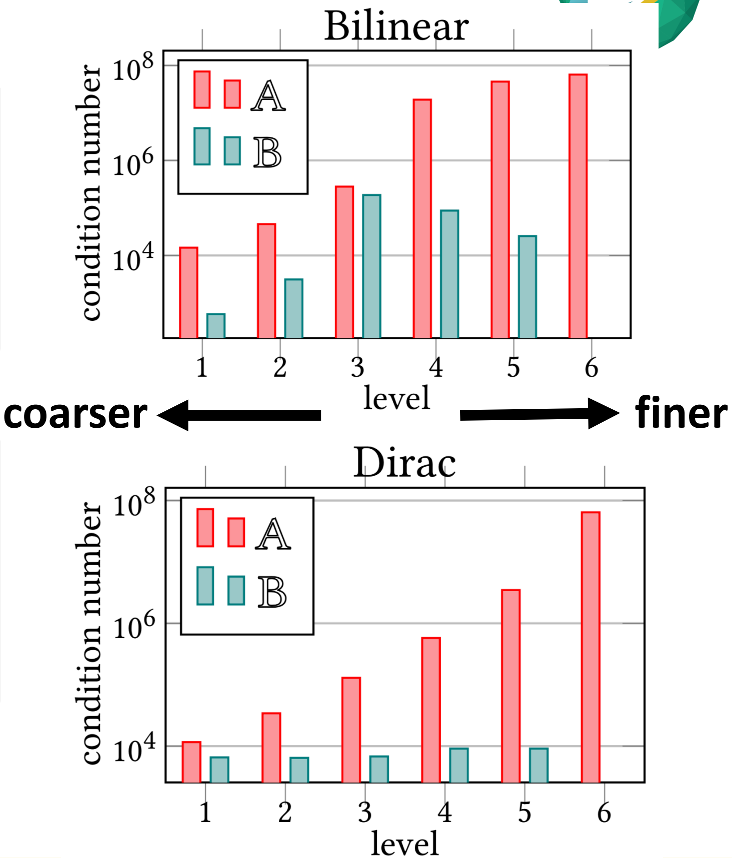




boundary aware

Spectrum and conditioning





Basis truncation

- The coarser the level is, the larger support region basis functions will have, which slows down
  - Matrix factorization
  - Matrix multiplication
- Fast decay property allows for truncation of the basis functions
  - See our paper for details
- Besides, geometrical invariance should be preserved

translation

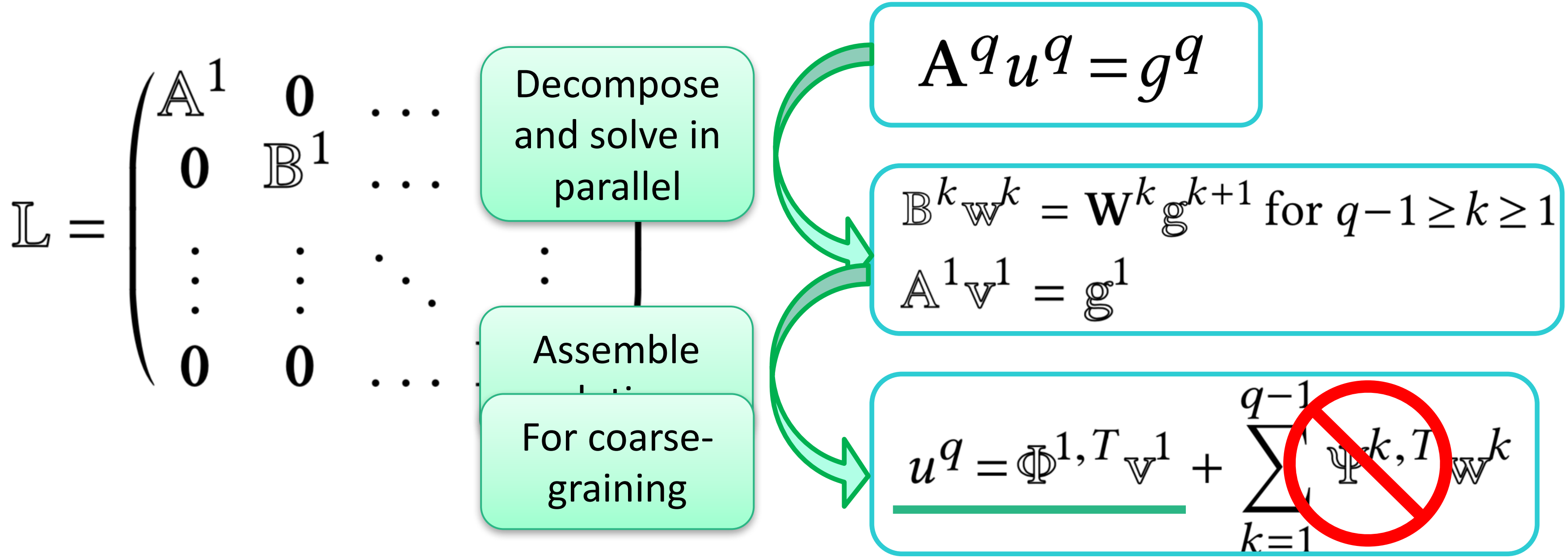
$$\forall j, \sum_i \mathbb{C}^k_{ij} = \mathbb{I}_{3\times 3},$$

Infinitesimal rotation

$$\forall j, \sum_i \mathbb{C}^k_{ij}[\bar{\mathbf{x}}^{k-1}_i]_\times = [\bar{\mathbf{x}}^k_j]_\times,$$

Multilevel solve

- Recall block diagonal stiffness matrix
- Therefore, each level can be solved independently



$$\mathbb{L} = \begin{pmatrix} \mathbb{A}^1 & \mathbf{0} & \cdots & \\ \mathbf{0} & \mathbb{B}^1 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \end{pmatrix}$$

Decompose and solve in parallel

Assemble

For coarse-graining

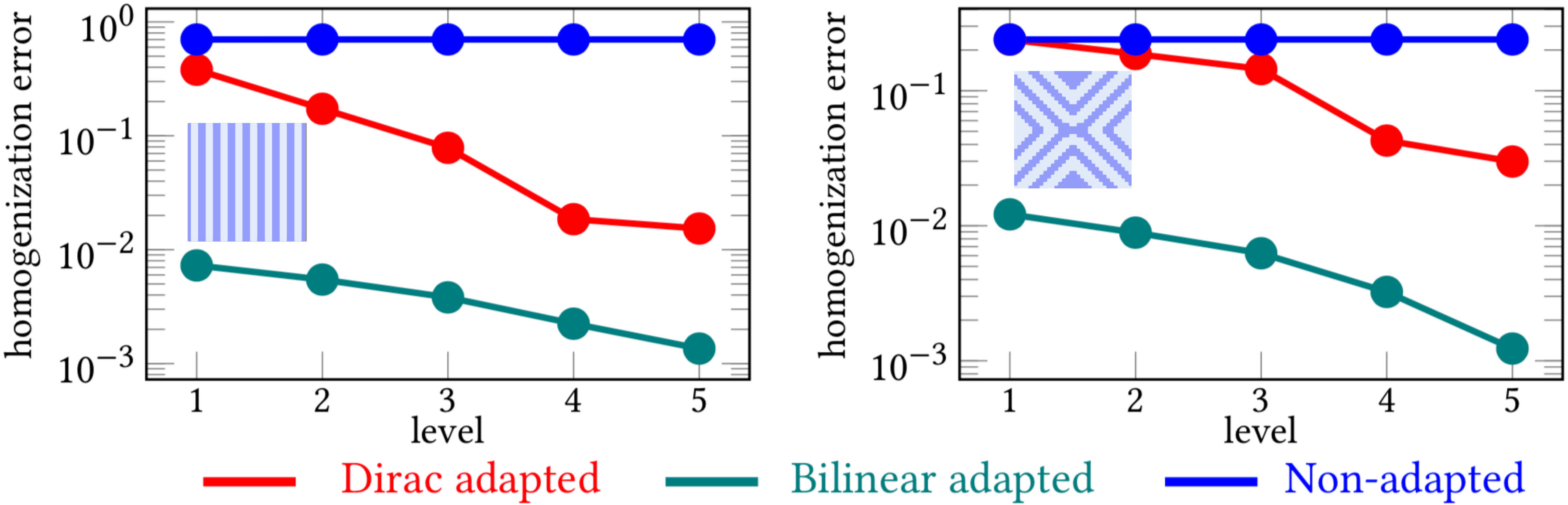
$$\mathbf{A}^q u^q = g^q$$

$$\mathbb{B}^k \mathbb{w}^k = \mathbf{W}^k \mathbb{g}^{k+1} \text{ for } q-1 \ge k \ge 1$$

$$\mathbb{A}^1 \mathbb{v}^1 = \mathbb{g}^1$$

$$\underline{u^q = \Phi^{1,T} \mathbb{v}^1} + \sum_{k=1}^{q-1} \cancel{\Psi^{k,T}} \mathbb{w}^k$$

Homogenization accuracy

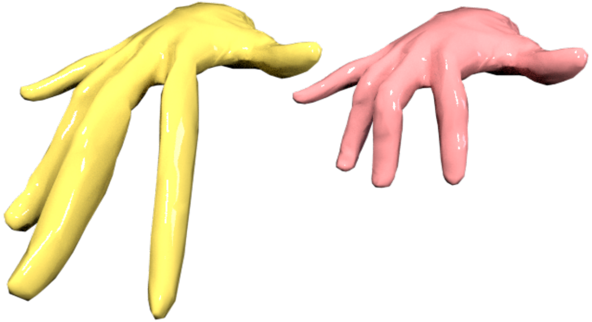


Resolve geometric nonlinearity

- Rotation-strain warping [Huang 2011]

$$\hat{u}^q = \arg\min_u \int_\Omega \|\nabla u - RS(\nabla u^q)\|_F^2$$

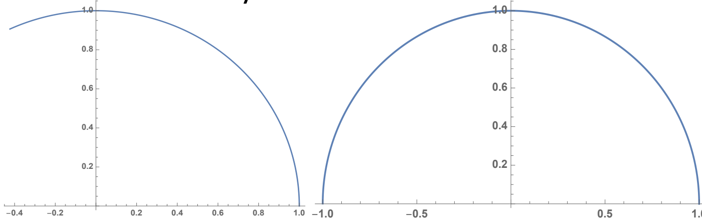
$$\text{s. t.} \quad S\hat{u}^q = 0$$



- Cayley mapping to reduce over-estimation of rotation by exp

$$\mathrm{Cay}(A) = (\mathbb{I}_{3\times3} - A/2)^{-1}(\mathbb{I}_{3\times3} + A/2)$$

$$f(\theta) = \frac{i + \theta/2}{i - \theta/2} \qquad f(\theta) = \exp(\mathrm{i}\theta)$$



$$\theta \in [0, \pi]$$

Results

Comparisons

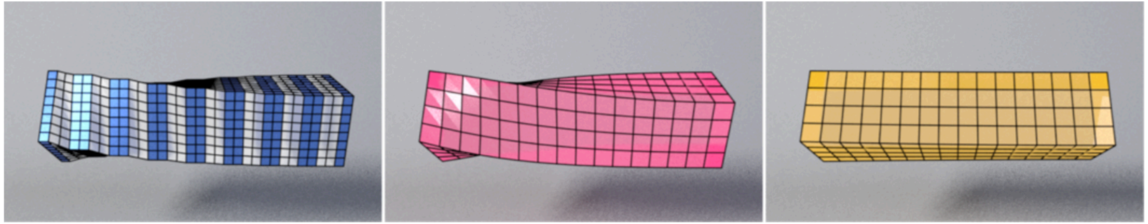
With [Nesme et al. 2009]

Element-wise diagonal matrix-valued basis functions

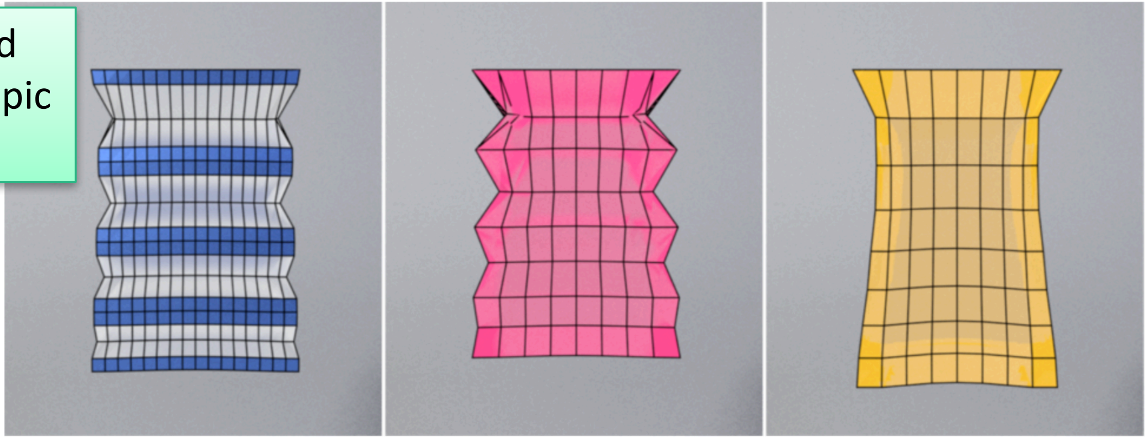
Both methods have only limited expressivity for complex anisotropic deformation.

With [Kharevych et al. 2009]

Regress elastic material tensor to encode anisotropy



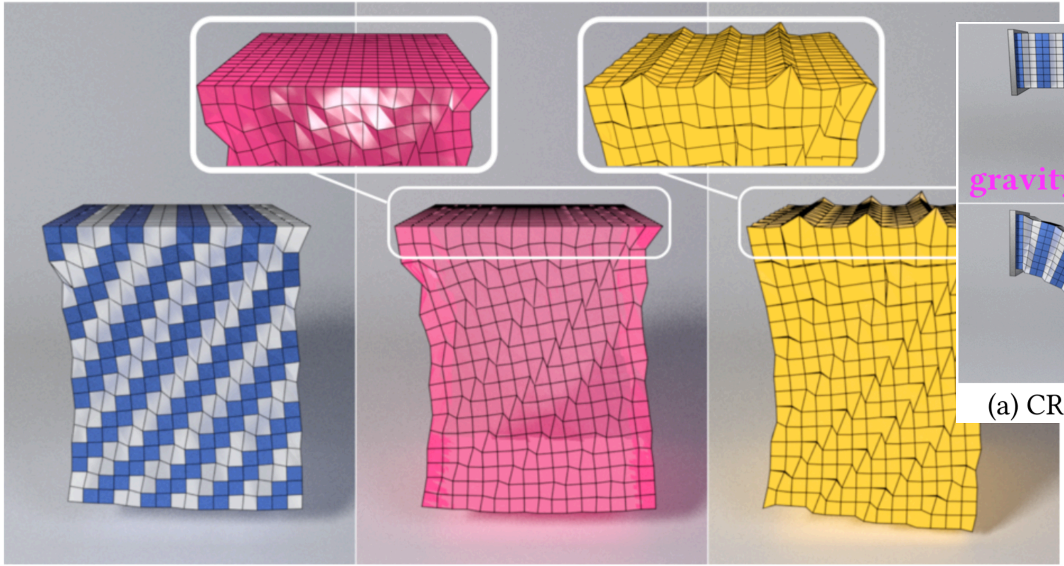
(a) CR groundtruth (b) our method (c) [Nesme et al. 2009]



(a) CR groundtruth (b) our method (c) [Kharevych et al. 2009]

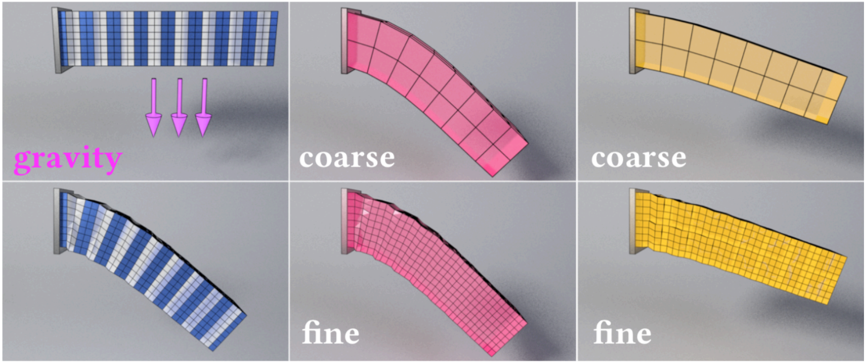
Comparisons

With [Chen 2018]



(a) CR groundtruth (b) our method (c) [Chen et al. 2018]

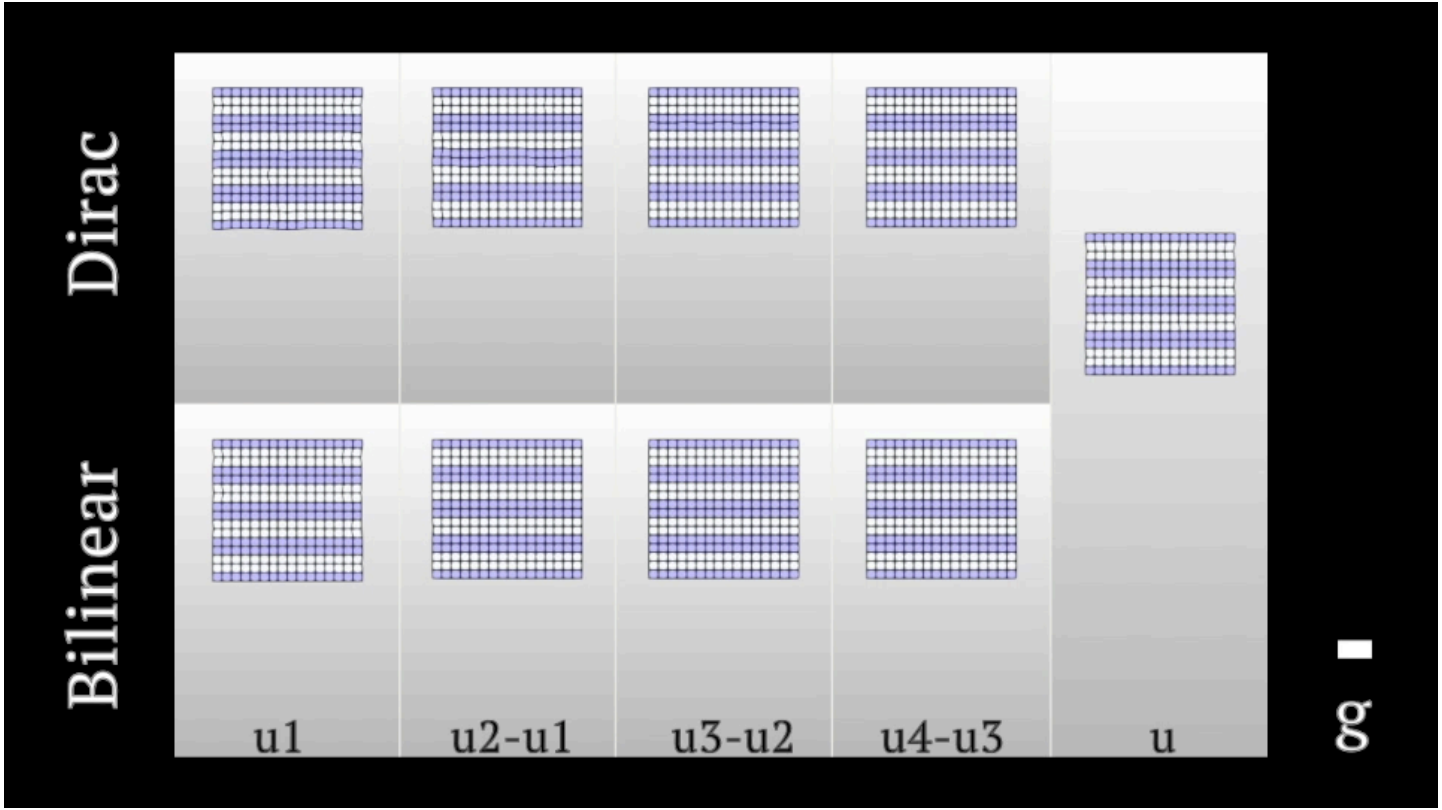
Cannot adapt to boundary conditions



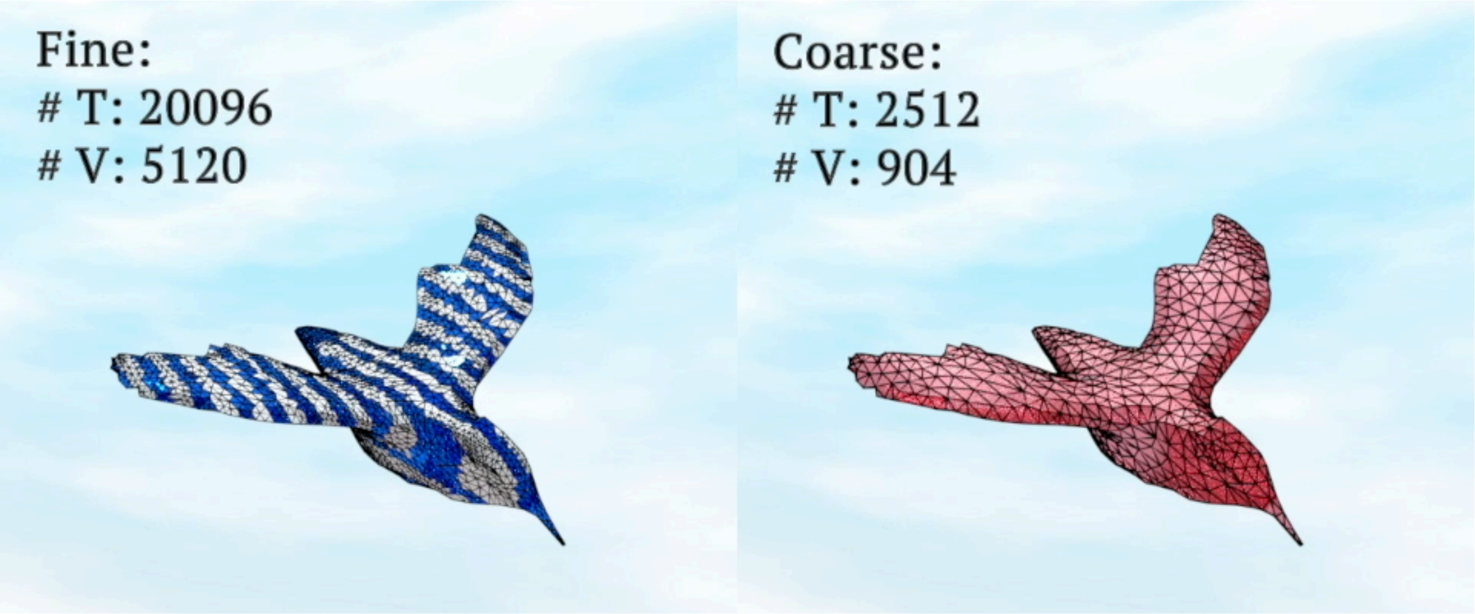
(a) CR groundtruth (b) our method (c) [Chen et al. 2018]

Cannot handle aggressive coarsening well

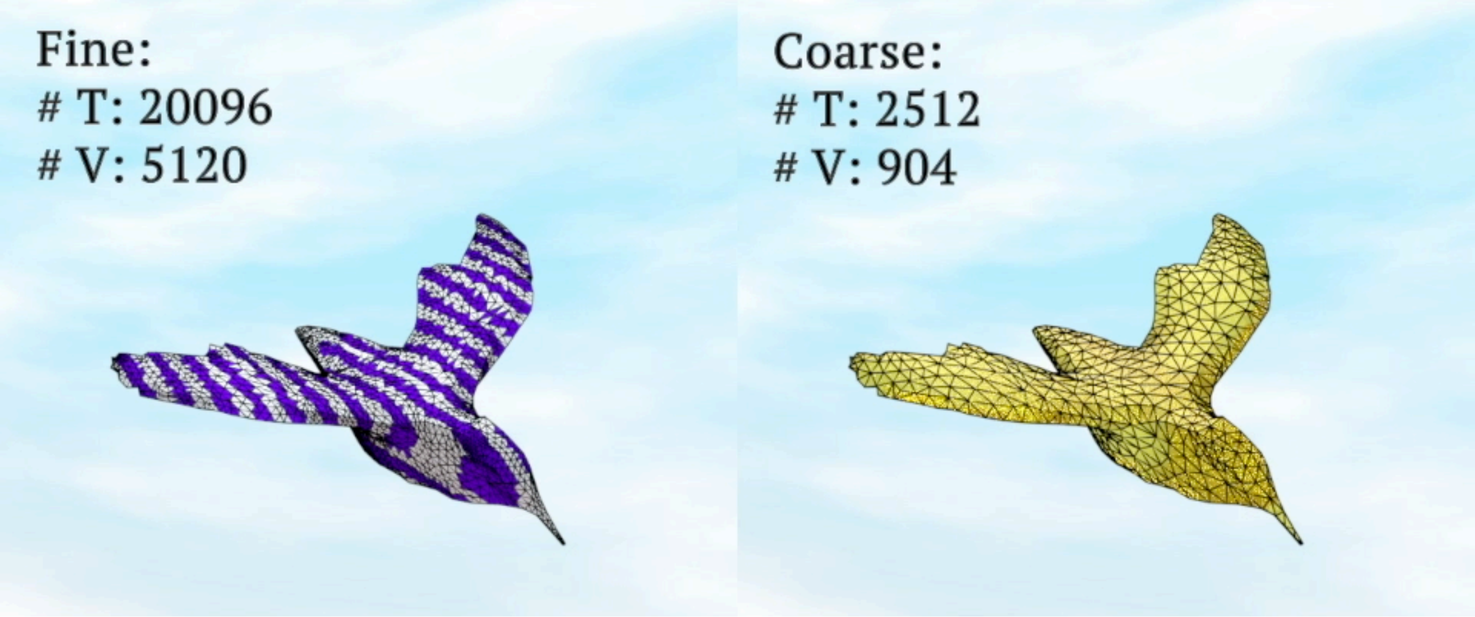
2D linear statics



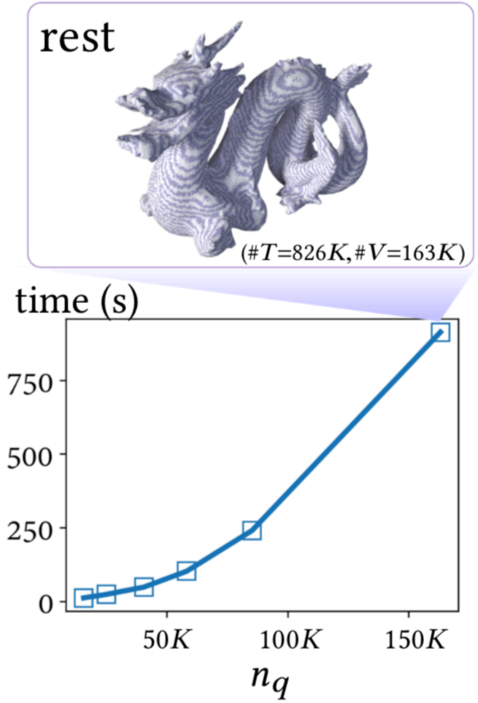
3D linear dynamics

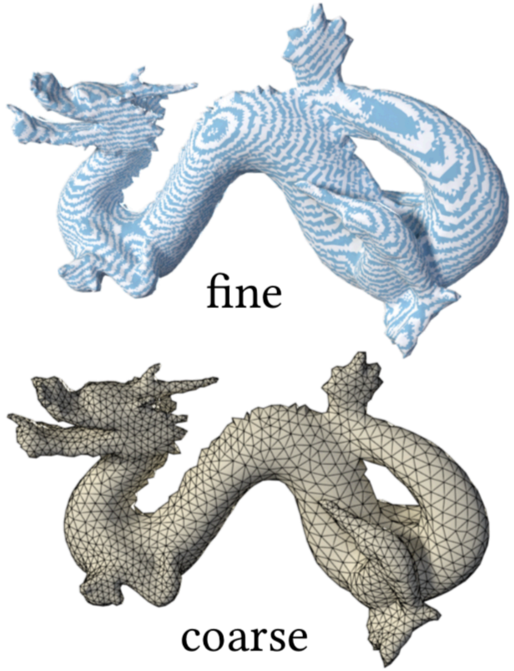


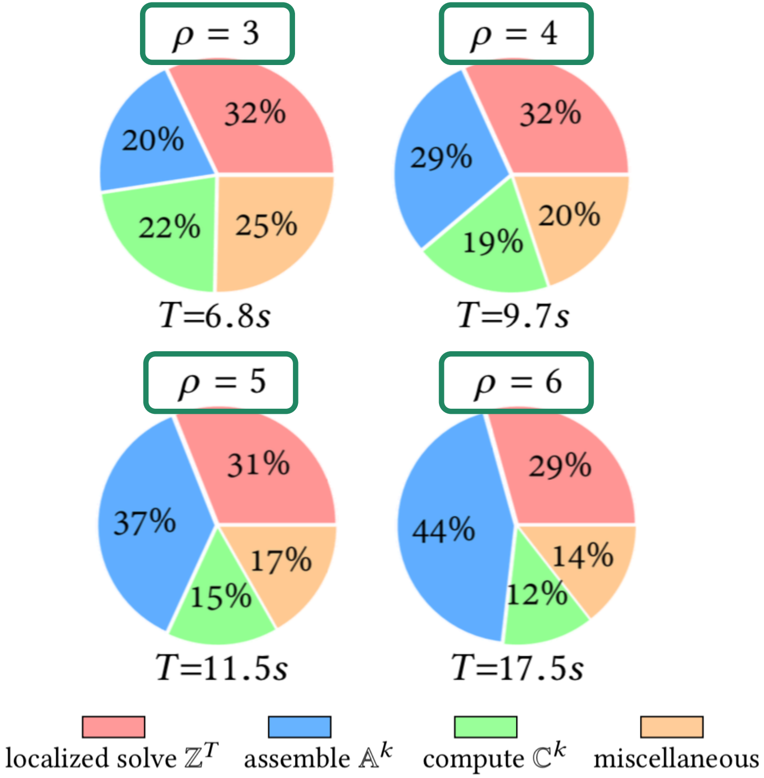
3D corotational dynamics



Complexity of precomputations



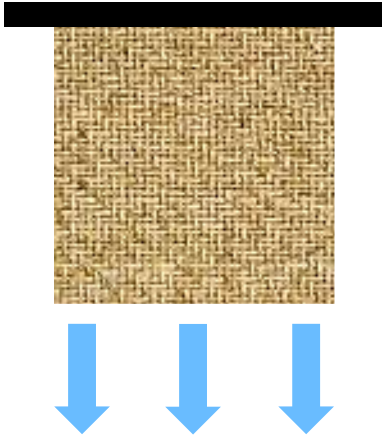


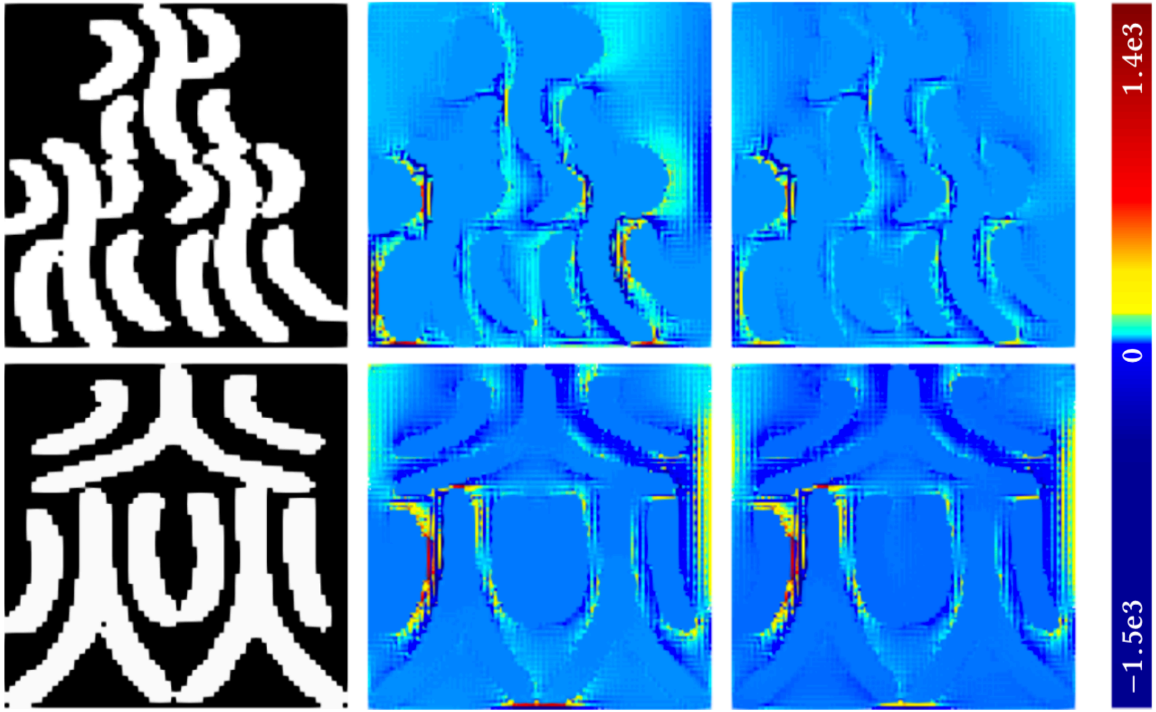


$O(n_q \log^{2d+1} n_q)$ [Owhadi 2017]

Stress homogenization

setting





composite material

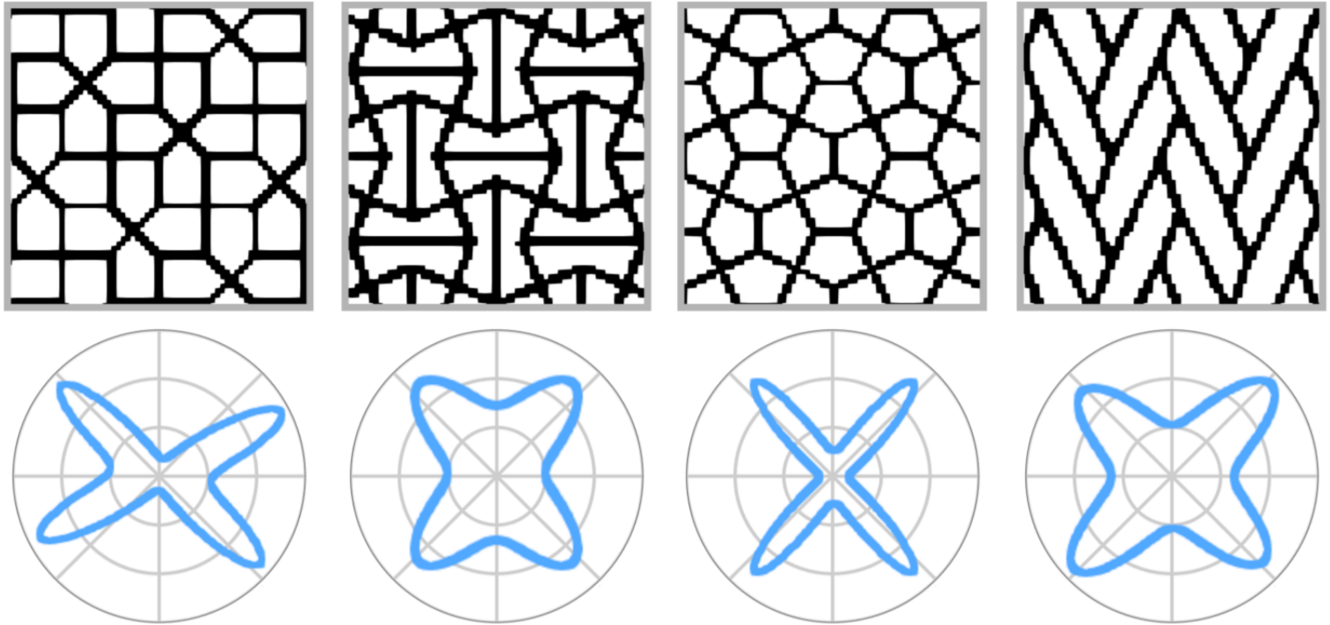
fine (64 × 64)

coarse (1 × 1)

Structure analysis

Tilling pattern

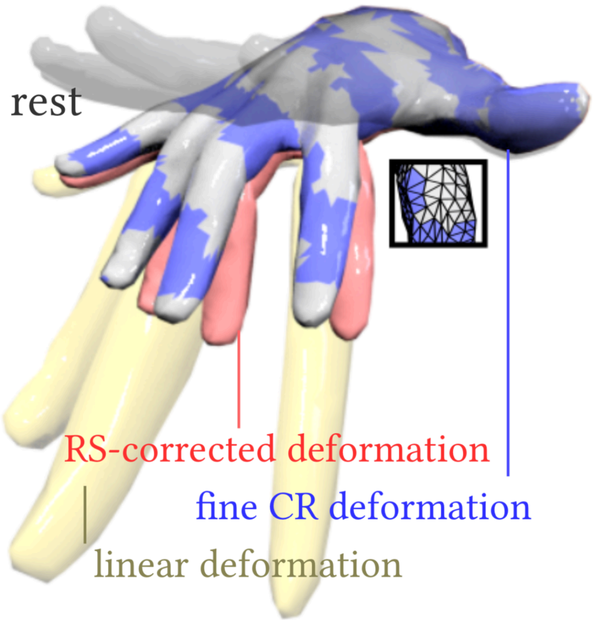
Directional Young's modulus



$$E(d) = 1/\mathrm{div}(dd^T) : (\mathbb{A}^1)^\dagger : \mathrm{div}(dd^T)$$

Limitations and future work

- More accurate handling of geometric non-linearity
- Push the efficiency to the limit
- Extend to general nonlinear problem
  - Analytically adapt the basis functions, or
  - Numerically adapt the basis via fast update
- Combined with CHARMS framework for local adaptation



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Thank you

Q&A

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