A Recursive Approach to Forward and Backward Substitutions

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1 Multilevel Coarse-graining

In [1], we constructed a hierarchy of material-adapted basis functions to coarsen high-resolution heterogeneous systems. The solution space was recursively decomposed into a subspace and its *operator-orthogonal* complement based on a closed-form solution. In our subsequent work [2], this multilevel construction was equivalently reformulated using a single Cholesky factorization to improve numerical performance. Once the original linear system is properly transformed with pre-wavelets, the coarsened operators can be directly extracted from the resulting Cholesky factor. This connection is established via the Schur complement. Specifically, for a given linear system $\mathbb{A}$, we first transform it to $\overline{\mathbf{A}}$ with degrees of freedom reordered from fine scale (denoted by $\mathtt{f}$) to coarse scale (denoted by $\mathtt{c}$). The system is then block-diagonalized as follows:

$$\underbrace{\begin{pmatrix} \mathbf{A}_{\mathtt{ff}} & \mathbf{A}_{\mathtt{fc}} \\ \mathbf{A}_{\mathtt{cf}} & \mathbf{A}_{\mathtt{cc}} \end{pmatrix}}_{\overline{\mathbf{A}}} = \underbrace{\begin{pmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{A}_{\mathtt{cf}}\mathbf{A}_{\mathtt{ff}}^{-1} & \mathbb{I} \end{pmatrix}}_{\overline{\mathbf{L}}} \underbrace{\begin{pmatrix} \mathbf{A}_{\mathtt{ff}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\mathtt{cc}} - \mathbf{A}_{\mathtt{cf}}\mathbf{A}_{\mathtt{ff}}^{-1}\mathbf{A}_{\mathtt{fc}} \end{pmatrix}}_{\overline{\mathbf{D}}} \underbrace{\begin{pmatrix} \mathbb{I} & \mathbf{A}_{\mathtt{ff}}^{-1}\mathbf{A}_{\mathtt{fc}} \\ \mathbf{0} & \mathbb{I} \end{pmatrix}}_{\overline{\mathbf{L}}^{\top}}.$$

Here, $\mathbf{A}_{\mathtt{ff}}$ is the Galerkin projection of $\mathbb{A}$ using wavelets, and its Schur complement $\mathbf{A}_{\mathtt{cc}} - \mathbf{A}_{\mathtt{cf}}\mathbf{A}_{\mathtt{ff}}^{-1}\mathbf{A}_{\mathtt{fc}}$ represents for the Galerkin projection of $\mathbb{A}$ using the *adapted* basis functions, namely

$$\mathbf{A}_{\mathtt{cc}} - \mathbf{A}_{\mathtt{cf}}\mathbf{A}_{\mathtt{ff}}^{-1}\mathbf{A}_{\mathtt{fc}} = \mathbb{C}\mathbb{A}\mathbb{C}^{\top}, \tag{1}$$

where $\mathbb{C}$ is the adapted basis refinement matrix defined in [1], separating the coarse scale from composite scales. By further factorizing the two diagonal blocks of $\overline{\mathbf{D}}$ using Cholesky decomposition, we can conclude that both can be directly assembled from $\overline{\mathbf{A}}$'s Cholesky factor, leveraging the properties of triangular matrices.

2 Substitution in a V-cycle

After rearranging all degrees of freedom into multiple scales, performing a Cholesky factorization decomposes the system into a hierarchical representation all at once, implicitly constructing the refinement operators $\mathbb{C}$ for coarse-graining fine-scale systems. These refinement operators can been seen as the restriction operators in the context of multigrid methods. In fact, forward and backward substitutions can also be computed recursively from fine to coarse scales, thereby conceptually imitating a multigrid solve. To demonstrate this, consider a lower triangular Cholesky factor consisting of two scales

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{\mathtt{ff}} & \mathbf{0} \\ \mathbf{L}_{\mathtt{cf}} & \mathbf{L}_{\mathtt{cc}} \end{pmatrix}, \tag{2}$$

as well as the right hand vector $\mathbf{g} = (\mathbf{g}_{\mathtt{f}}, \mathbf{g}_{\mathtt{c}})^\top$ and the solution vector $\mathbf{u} = (\mathbf{u}_{\mathtt{f}}, \mathbf{u}_{\mathtt{c}})^\top$. In Alg. 1, we show how to convert solving $\mathbf{L}\mathbf{L}^\top\mathbf{u} = \mathbf{g}$ into a recursive, "V-cycle"-like scheme. This analogy may inspire new ideas for designing multigrid operators.

Input: Cholesky factor $\mathbf{L}^k$ s.t. $\mathbf{A}^k = \mathbf{L}^k\mathbf{L}^{k,\top}$, right-hand side vector $\mathbf{g}^k$
Output: Solution $\mathbf{u}^k$ to $\mathbf{A}^k\mathbf{u}^k = \mathbf{g}^k$
Subroutine `Vcycle(`$\mathbf{L}^k, \mathbf{g}^k, \mathbf{u}^k$`)`
  if $k == \mathit{coarsest\_level}$ **then**
    solve $\mathbf{L}^k \left(\mathbf{L}^k\right)^\top \mathbf{u}^k = \mathbf{g}^k$;
  else
    $\mathbf{r}^k_{\mathtt{f}} \leftarrow \left(\mathbf{L}^k_{\mathtt{ff}}\right)^{-1} \mathbf{g}^k_{\mathtt{f}}$; `/* pre-relaxation */`
    $\mathbf{g}^k_{\mathtt{c}} \leftarrow \mathbf{g}^k_{\mathtt{c}} - \mathbf{L}^k_{\mathtt{cf}}\mathbf{r}^k_{\mathtt{f}}$; `/* restriction */`
    `Vcycle(`$\mathbf{L}^{k-1} = \mathbf{L}^k_{\mathtt{c}}, \mathbf{g}^{k-1} = \mathbf{g}^k_{\mathtt{c}}, \mathbf{u}^{k-1} = \mathbf{u}^k_{\mathtt{c}}$`)`; `/* next level */`
    $\mathbf{g}^k_{\mathtt{f}} \leftarrow \mathbf{r}^k_{\mathtt{f}} - \left(\mathbf{L}^k_{\mathtt{cf}}\right)^\top \mathbf{u}^k_{\mathtt{c}}$; `/* prolongation */`
    $\mathbf{u}^k_{\mathtt{f}} \leftarrow \left(\mathbf{L}^k_{\mathtt{ff}}\right)^{-\top} \mathbf{g}^k_{\mathtt{f}}$; `/* post-relaxation */`
  end

Algorithm 1: A V-cycle reformulation of substitutions.

References

[1] Chen, J., Budninskiy, M., Owhadi, H., Bao, H., Huang, J., and Desbrun, M. Material-adapted refinable basis functions for elasticity simulation. *ACM Transactions on Graphics (TOG) 38*, 6 (2019), 1–15.

[2] Chen, J., Schäfer, F., Huang, J., and Desbrun, M. Multiscale cholesky preconditioning for ill-conditioned problems. *ACM Transactions on Graphics (TOG) 40*, 4 (2021), 1–13.