Lightning-fast Method of Fundamental Solutions

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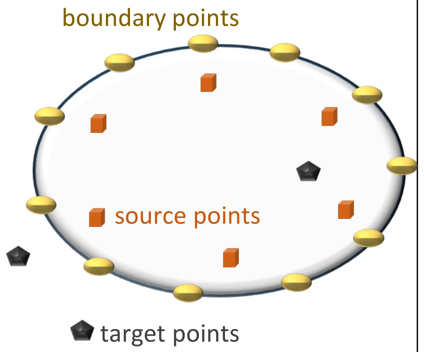
Two stages of the boundary element method

Solve stage: solve for "charges" that enforce the given boundary conditions

$$\int_{\mathcal{M}} G(\boldsymbol{z},\boldsymbol{y})\sigma(\boldsymbol{y})\,\mathrm{d}v_{\boldsymbol{y}} = b(\boldsymbol{z}) \quad \forall \boldsymbol{z} \in \mathcal{M}.$$

Evaluation stage: evaluate the resulting "potential" at any target point in space

$$u(\boldsymbol{x}) = \int_{\mathcal{M}} G(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{y})\,\mathrm{d}v_{\boldsymbol{y}}.$$



Discretization of the Boundary Integral Equation

$$\underbrace{\sum_{j=1}^{S}\left(\iint_{\mathcal{M}\times\mathcal{M}} \phi_i(\boldsymbol{y})G(\boldsymbol{y},\boldsymbol{z})\psi_j(\boldsymbol{z})\,\mathrm{d}v_{\boldsymbol{y}}\mathrm{d}v_{\boldsymbol{z}}\right)}_{\boldsymbol{K}} \cdot \underbrace{s_j}_{\boldsymbol{s}} = \underbrace{\int_{\mathcal{M}} b(\boldsymbol{y})\phi_i(\boldsymbol{y})\mathrm{d}v_{\boldsymbol{y}}}_{\boldsymbol{b}}$$

Method of fundamental solution

If basis functions are Dirac delta functions, discrete BIE reduces to

$$\sum_{j=1}^{S} G(\boldsymbol{y}_i,\boldsymbol{z}_j)\, s_j = b_i. \quad \forall i = 1..B$$

Inverse Cholesky preconditioner

BIE/MFS matrices are conceptually similar to the inverse of their corresponding differential operator

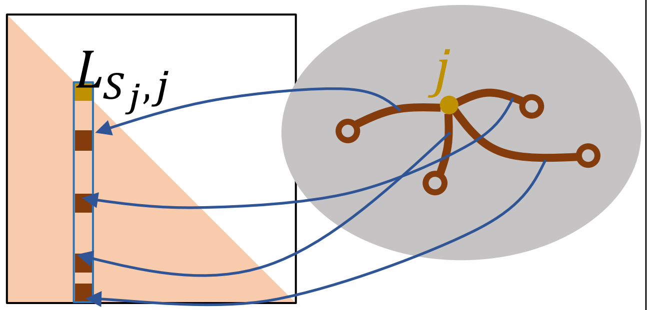
$$Ks = b \;\rightarrow\; K^{-1} \approx L_S L_S^T \;\rightarrow\; s \approx L_S L_S^T b$$

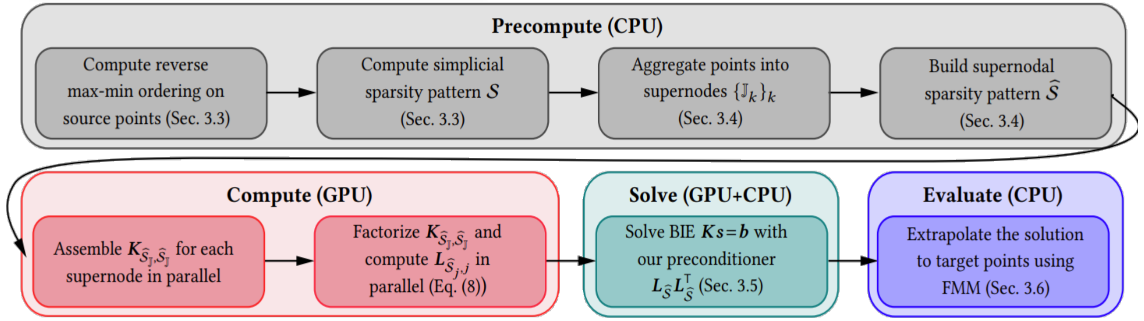
Kaporin's construction of the inverse Cholesky factor

$$L_{S_j,j} = \frac{K^{-1}_{S_j,S_j}\mathbb{e}_j}{\sqrt{\mathbb{e}_j^T K^{-1}_{S_j,S_j}\mathbb{e}_j}}, \quad \forall j = 1..B,$$ [Kaporin 1994]

Minimizer of Kaporin condition number

$$\kappa_{\mathrm{Kap}} = \frac{1}{B}\frac{\mathrm{tr}(\boldsymbol{K}\boldsymbol{L}_S\boldsymbol{L}_S^T)}{\det(\boldsymbol{K}\boldsymbol{L}_S\boldsymbol{L}_S^T)^{\frac{1}{B}}}$$



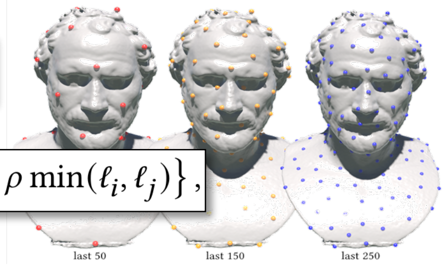


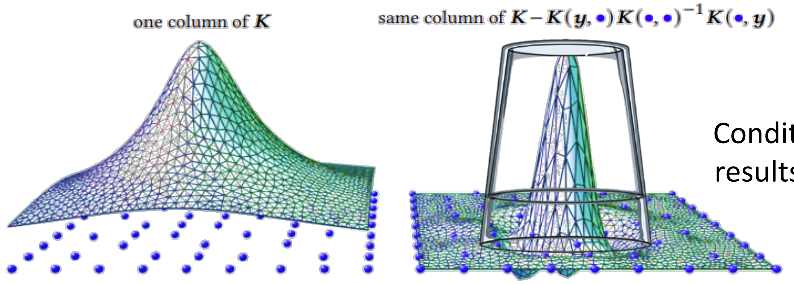
Ordering

$$i_k = \operatorname*{argmax}_{q} \min_{p\in\{0,k-1\}} \mathrm{dist}(\boldsymbol{y}_q, \boldsymbol{y}_{i_p})$$

Sparsity pattern

$$\mathcal{S} := \{(i,j) | i \geq j \text{ and } \mathrm{dist}(x_i,x_j) \leq \rho \min(\ell_i,\ell_j)\},$$





Screening effect:
Conditioning a subset of points results in localized correlations

Applications