Go Green: General Regularized Green's Functions for Elasticity — Supplemental Material

JIONG CHEN, LTCI, Telecom Paris, Institut Polytechnique de Paris, France
MATHIEU DESBRUN, Inria / Ecole Polytechnique, France

A VOIGT NOTATION

For linear elastic materials, the relation between the stress $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\epsilon}$ is given by Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\epsilon}, \quad \text{or equivalently} \quad \boldsymbol{\epsilon} = \mathbf{S} : \boldsymbol{\sigma},$$

where $\mathbf{C}$ is the fourth-order elasticity tensor and $\mathbf{S}$ is its inverse, called the *compliance tensor.* Due to its major and minor symmetries, the tensor $\mathbf{C}$ only has 21 independent values. In Voigt notation, Hooke's law can be expressed in matrix form, i.e., $\boldsymbol{\sigma}^V = \mathbf{C}^V \boldsymbol{\epsilon}^V$ with

$$\boldsymbol{\sigma}^V = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}, \boldsymbol{\epsilon}^V = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}, \mathbf{C}^V = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix}.$$

For *isotropic* materials, the independent 21 coefficients reduce to two scalars, often parameterized by Young's modulus E and Poisson ratio ν. For *orthotropic* materials, the material tensor has 9 independent coefficients involving 3 Young's moduli, 3 Poisson ratios, and 3 shear moduli, with a symmetric compliance matrix written as

$$\mathbf{S}^V_{\text{orth}} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix},$$

where E_i is the Young's modulus in direction $\mathbf{e}_i$, ν_{ij} is the Poisson ratio encoding the rate of contraction along $\mathbf{e}_j$ for an extension along $\mathbf{e}_i$, and G_{ij} is the shear stiffness for the plane spanned by $\mathbf{e}_i$ and $\mathbf{e}_j$.

B SOLUTION FOR ISOTROPIC MATERIALS

For isotropic elastic materials, the elasticity tensor C_{ijkl} only has two degrees of freedoms, the Lamé coefficients μ and λ. For such an isotropic elastic material, and if we consider a singular load (i.e., g_ε taken to be a Diract delta function), the radial term $R_l(r)$ in the Green's function reduces to

$$\int_0^\infty j_l(|\mathbf{x}||\boldsymbol{\xi}|)\,\mathrm{d}|\boldsymbol{\xi}| = \frac{\sqrt{\pi}\Gamma((l+1)/2)}{2|\mathbf{x}|\Gamma((l+2)/2)},$$

which is singular at the origin $\mathbf{x}=\mathbf{0}$. The overall expression of the Green's function is assembled from the only two non-vanishing degrees ($l=0$ and $l=2$), yielding

$$G_{11}(r,\theta,\varphi) = \frac{\sin^2(\theta)(\lambda+\mu)\cos^2(\varphi)+\lambda+3\mu}{8\pi\mu r(\lambda+2\mu)},$$
$$G_{12}(r,\theta,\varphi) = G_{21}(r,\theta,\varphi) = \frac{\sin^2(\theta)(\lambda+\mu)\sin(\varphi)\cos(\varphi)}{8\pi\mu r(\lambda+2\mu)},$$
$$G_{13}(r,\theta,\varphi) = G_{31}(r,\theta,\varphi) = \frac{\sin(\theta)\cos(\theta)(\lambda+\mu)\cos(\varphi)}{8\pi\mu r(\lambda+2\mu)},$$
$$G_{22}(r,\theta,\varphi) = \frac{\sin^2(\theta)(\lambda+\mu)\sin^2(\varphi)+\lambda+3\mu}{8\pi\mu r(\lambda+2\mu)},$$
$$G_{23}(r,\theta,\varphi) = G_{32}(r,\theta,\varphi) = \frac{\sin(\theta)\cos(\theta)(\lambda+\mu)\sin(\varphi)}{8\pi\mu r(\lambda+2\mu)},$$
$$G_{33}(r,\theta,\varphi) = \frac{\lambda\cos^2(\theta)+\mu\cos^2(\theta)+\lambda+3\mu}{8\pi\mu r(\lambda+2\mu)}.$$

It is obvious that $\mathbf{G}$ is singular at the origin due to the non-regularized load, and one can easily verify that this corresponds to the Kelvin solution for isotropic linear elasticity given in [Cortez et al. 2005; de Goes and James 2017] — namely, in Euclidean coordinates:

$$\mathbf{u}(\mathbf{x}) = \left[\frac{(a-b)}{|\mathbf{x}|}\mathbb{I}_3 + \frac{b}{|\mathbf{x}|^3}\mathbf{x}\mathbf{x}^\top\right]\mathbf{f} \equiv \mathbf{G}(\mathbf{x})\mathbf{f},$$

where $a=1/(4\pi\mu)$, $b=a/(4(1-\nu))$, and $\nu=\lambda/(2(\mu+\lambda))$. When applying a smooth load, for instance $g_\varepsilon(r)=15\varepsilon^4/(8\pi)(r^2+\varepsilon^2)^{-\frac{7}{2}}$, whose Fourier transform is

$$\widehat{g_\varepsilon}(|\boldsymbol{\xi}|) = \frac{\varepsilon^2|\boldsymbol{\xi}|^2 K_2(\varepsilon|\boldsymbol{\xi}|)}{2}$$

and K_α is the modified Bessel function of the second kind, the radial term in our Green's function becomes

$$R_l(r) = \frac{\sqrt{\pi}}{2} r^l \varepsilon^{-l-1}\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(\frac{l+5}{2}\right){}_2\widetilde{F}_1\left(\frac{l+1}{2},\frac{l+5}{2};l+\frac{3}{2};-\frac{r^2}{\varepsilon^2}\right),$$

where ${}_2\widetilde{F}_1$ is the regularized hypergeometric function. This integral, when evaluated for degree 0 and 2, simplifies to:

$$R_0(r) = \frac{\pi\left(2r^2+3\epsilon^2\right)}{4\left(r^2+\epsilon^2\right)^{3/2}}, \quad R_2(r) = \frac{\pi r^2}{4\left(r^2+\epsilon^2\right)^{3/2}},$$

where the singularities at $r=0$ has now disappeared. Now with this regularized radial term, the Green's function becomes:

$$G_{11}(r,\theta,\varphi) = \frac{r^2\sin^2(\theta)(\lambda+\mu)\cos^2(\varphi)+r^2(\lambda+3\mu)+\varepsilon^2(2\lambda+5\mu)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},$$
$$G_{12}(r,\theta,\varphi) = G_{21}(r,\theta,\varphi) = \frac{r^2\sin^2(\theta)(\lambda+\mu)\sin(\varphi)\cos(\varphi)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},$$
$$G_{13}(r,\theta,\varphi) = G_{31}(r,\theta,\varphi) = \frac{r^2\sin(\theta)\cos(\theta)(\lambda+\mu)\cos(\varphi)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},$$

Authors' addresses: J. Chen, Telecom Paris (IP Paris), 19 Place Marguerite Perey, 91120 Palaiseau, France; M. Desbrun, Inria Saclay/Ecole Polytechnique (IP Paris), 1 rue Honoré d'Estienne d'Orves, 91120 Palaiseau, France.

$$G_{22}(r,\theta,\varphi) = \frac{r^2\sin^2(\theta)(\lambda+\mu)\sin^2(\varphi) + r^2(\lambda+3\mu) + \varepsilon^2(2\lambda+5\mu)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},$$

$$G_{23}(r,\theta,\varphi) = G_{32}(r,\theta,\varphi) = \frac{r^2\sin(\theta)\cos(\theta)(\lambda+\mu)\sin(\varphi)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},$$

$$G_{33}(r,\theta,\varphi) = \frac{r^2\cos^2(\theta)(\lambda+\mu) + r^2(\lambda+3\mu) + \varepsilon^2(2\lambda+5\mu)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}}.$$

One can check analytically that this exactly reproduces the regularized Kelvin solution proposed in [de Goes and James 2017].

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