GO GREEN: GENERAL REGULARIZED GREEN'S FUNCTIONS FOR ELASTICITY

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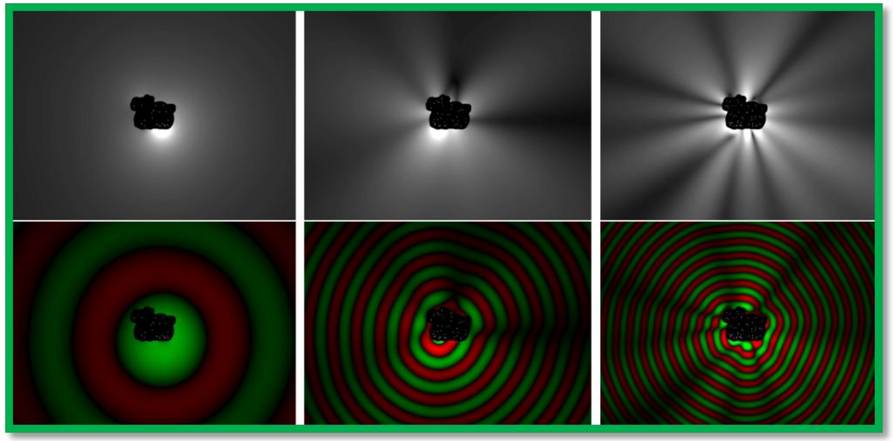
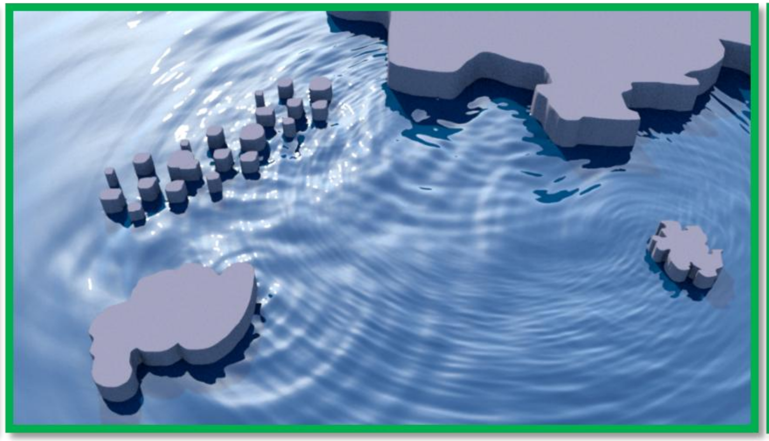
GREEN'S FUNCTION OF PDES

Green's function – **analytical** solution to **homogenous** & **"boundless"** PDEs *w.r.t.* a **singular** impulse

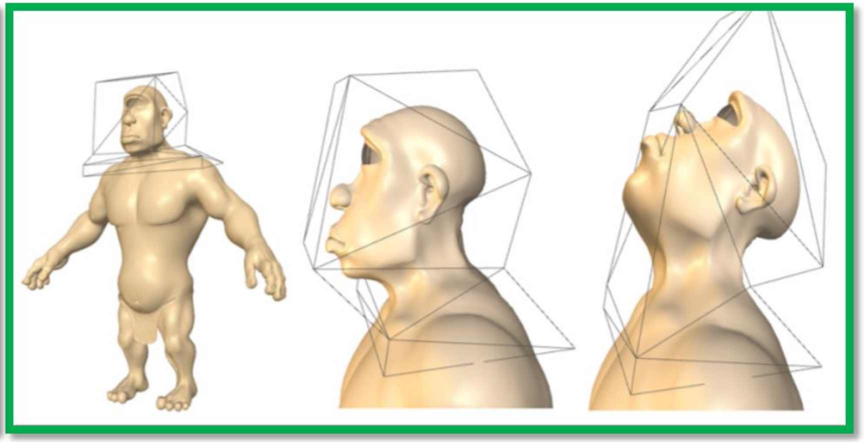
- Used for real-time simulation
- Foundation of the boundary element method
- PDE types: wave, Poisson, Helmholtz…



Sound synthesis [James et al. 2006]



Wave animation [Schreck et al. 2019]



Cage deformation [Lipman et al. 2008]

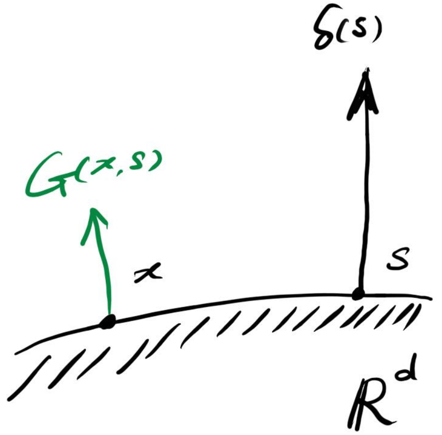
METHOD OF GREEN'S FUNCTIONS

- Given a linear and homogeneous PDE

$$\mathcal{L}u(x) = f(x)$$

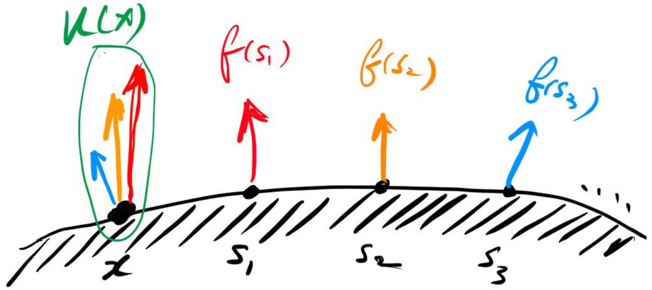
- A Green's function $G(x,s)$ is defined as

$$\mathcal{L}G(x,s) = \delta(x-s)$$



- Solution expressed via convolution

$$u(x) = \int G(x,s) f(s)\,\mathrm{d}s$$

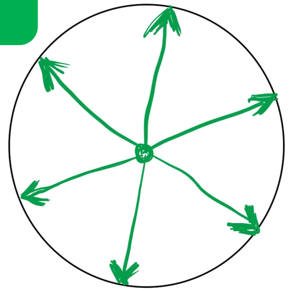


No equation solves, cheap to evaluate!

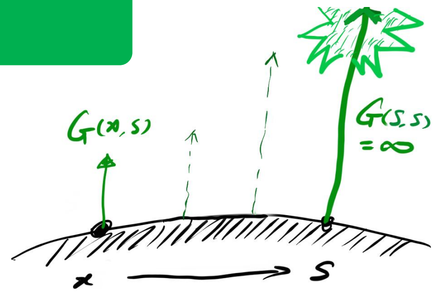
FAMILY OF GREEN'S FUNCTIONS

Differential operator L	Green's function G	Example of application
∂_t^{n+1}	$\frac{t^n}{n!}\Theta(t)$	
$\partial_t+\gamma$	$\Theta(t)e^{-\gamma t}$	
$(\partial_t+\gamma)^2$	$\Theta(t)te^{-\gamma t}$	
$\partial_t^2+2\gamma\partial_t+\omega_0^2$ where $\gamma<\omega_0$	$\Theta(t)e^{-\gamma t}\frac{\sin(\omega t)}{\omega}$ with $\omega=\sqrt{\omega_0^2-\gamma^2}$	1D underd
$\partial_t^2+2\gamma\partial_t+\omega_0^2$ where $\gamma>\omega_0$	$\Theta(t)e^{-\gamma t}\frac{\sinh(\omega t)}{\omega}$ with $\omega=\sqrt{\gamma^2-\omega_0^2}$	1D overdar
$\partial_t^2+2\gamma\partial_t+\omega_0^2$ where $\gamma=\omega_0$	$\Theta(t)e^{-\gamma t}t$	1D criticall
2D Laplace operator $\nabla^2_{\text{2D}}=\partial_x^2+\partial_y^2$	$\frac{1}{2\pi}\ln\rho$ with $\rho=\sqrt{x^2+y^2}$	2D Poisson
3D Laplace operator $\nabla^2_{\text{3D}}=\partial_x^2+\partial_y^2+\partial_z^2$	$\frac{-1}{4\pi r}$ with $r=\sqrt{x^2+y^2+z^2}$	Poisson equation
Helmholtz operator $\nabla^2_{\text{3D}}+k^2$	$\frac{-e^{-ikr}}{4\pi r}=i\sqrt{\frac{k}{32\pi r}}H^{(2)}_{1/2}(kr)=i\frac{k}{4\pi}h^{(2)}_0(kr)$	stationary 3D Schrödinger equation for free particle
∇^2-k^2 in n dimensions	$-(2\pi)^{-n/2}\left(\frac{k}{r}\right)^{n/2-1}K_{n/2-1}(kr)$	Yukawa potential, Feynman propagator
$\partial_t^2-c^2\partial_x^2$	$\frac{1}{2c}\Theta(t-\lvert x/c\rvert)$	1D wave equation
$\partial_t^2-c^2\nabla^2_{\text{2D}}$	$\frac{1}{2\pi c\sqrt{c^2t^2-\rho^2}}\Theta(t-\rho/c)$	2D wave equation
D'Alembert operator $\square=\frac{1}{c^2}\partial_t^2-\nabla^2_{\text{3D}}$	$\frac{\delta(t-\frac{r}{c})}{4\pi r}$	3D wave equation
$\partial_t-k\partial_x^2$	$\Theta(t)\left(\frac{1}{4\pi kt}\right)^{1/2}e^{-x^2/4kt}$	1D diffusion
$\partial_t-k\nabla^2_{\text{2D}}$	$\Theta(t)\left(\frac{1}{4\pi kt}\right)e^{-\rho^2/4kt}$	2D diffusio
$\partial_t-k\nabla^2_{\text{3D}}$	$\Theta(t)\left(\frac{1}{4\pi kt}\right)^{3/2}e^{-r^2/4kt}$	3D diffusio
$\frac{1}{c^2}\partial_t^2-\partial_x^2+\mu^2$	$\frac{1}{2}\left[(1-\sin\mu ct)(\delta(ct-x)+\delta(ct+x))+\mu\Theta(ct-\lvert x\rvert)J_0(\mu u)\right]$ with $u=\sqrt{c^2t^2-x^2}$	1D Klein–G
$\frac{1}{c^2}\partial_t^2-\nabla^2_{\text{2D}}+\mu^2$	$\frac{1}{4\pi}\left[(1+\cos(\mu ct))\frac{\delta(ct-\rho)}{\rho}+\mu^2\Theta(ct-\rho)\operatorname{sinc}(\mu u)\right]$ with $u=\sqrt{c^2t^2-\rho^2}$	2D Klein–Gordon equation
$\square+\mu^2$	$\frac{1}{4\pi}\left[\frac{\delta\left(t-\frac{r}{c}\right)}{r}+\mu c\Theta(ct-r)\frac{J_1(\mu u)}{u}\right]$ with $u=\sqrt{c^2t^2-r^2}$	3D Klein–Gordon equation
$\partial_t^2+2\gamma\partial_t-c^2\partial_x^2$	$\frac{1}{2}e^{-\gamma t}\left[\delta(ct-x)+\delta(ct+x)+\Theta(ct-\lvert x\rvert)\left(\frac{\gamma}{c}I_0\left(\frac{\gamma u}{c}\right)+\frac{\gamma t}{u}I_1\left(\frac{\gamma u}{c}\right)\right)\right]$ with $u=\sqrt{c^2t^2-x^2}$	telegrapher's equation
$\partial_t^2+2\gamma\partial_t-c^2\nabla^2_{\text{2D}}$	$\frac{e^{-\gamma t}}{4\pi}\left[(1+e^{-\gamma t}+3\gamma t)\frac{\delta(ct-\rho)}{\rho}+\Theta(ct-\rho)\left(\frac{\gamma\sinh\left(\frac{\gamma u}{c}\right)}{cu}+\frac{3\gamma t\cosh\left(\frac{\gamma u}{c}\right)}{u^2}-\frac{3ct\sinh\left(\frac{\gamma u}{c}\right)}{u^3}\right)\right]$ with $u=\sqrt{c^2t^2-\rho^2}$	2D relativistic heat conduction
$\partial_t^2+2\gamma\partial_t-c^2\nabla^2_{\text{3D}}$	$\frac{e^{-\gamma t}}{20\pi}\left[(8-3e^{-\gamma t}+2\gamma t+4\gamma^2t^2)\frac{\delta(ct-r)}{r^2}+\frac{\gamma^2}{c}\Theta(ct-r)\left(\frac{1}{cu}I_1\left(\frac{\gamma u}{c}\right)+\frac{4t}{u^2}I_2\left(\frac{\gamma u}{c}\right)\right)\right]$ with $u=\sqrt{c^2t^2-r^2}$	3D relativistic heat conduction

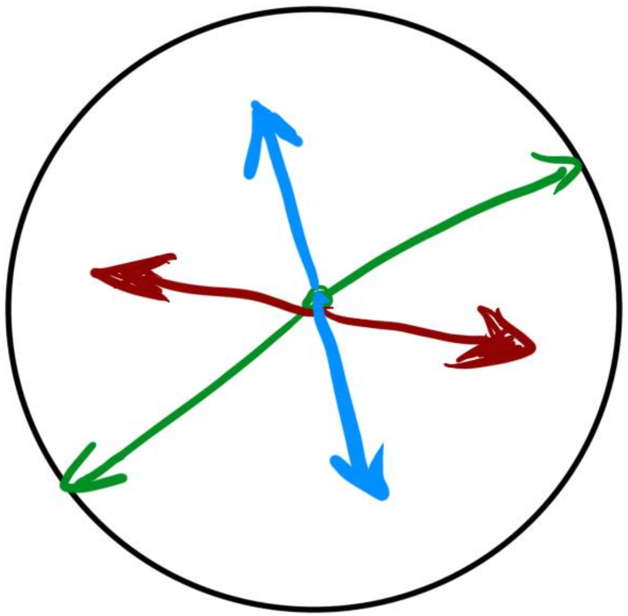
PROBLEM I: Operators & GFs are **isotropic**: they behave **the same in all directions.**



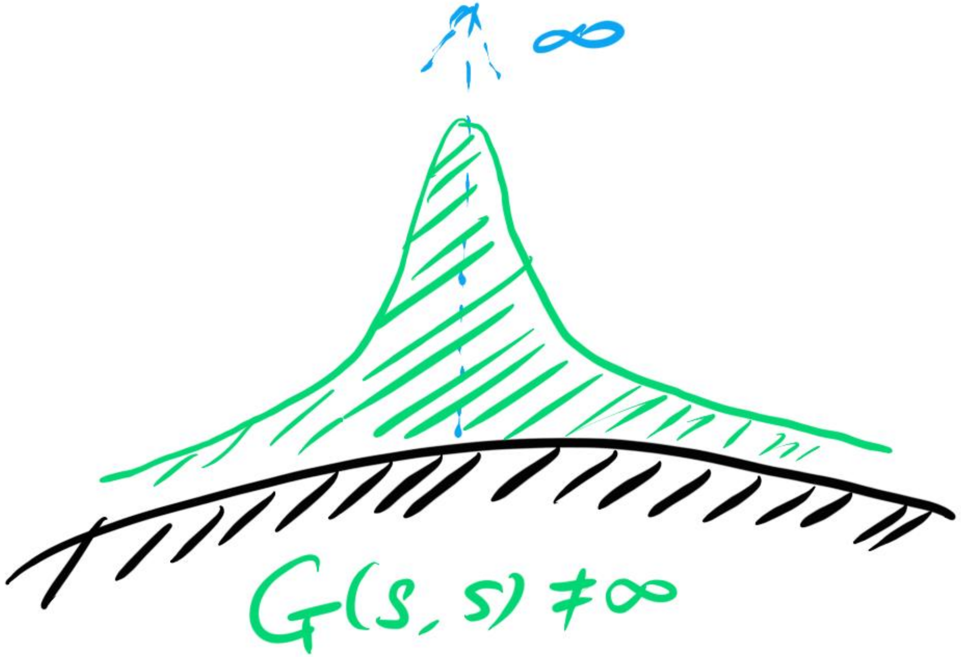
PROBLEM II: GFs are **singular** at the impulse.



Extend Green's function to support…



Anisotropy



Arbitrary regularization

ELASTICITY EQUATION

- Isotropic strain-stress relationship

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}$$

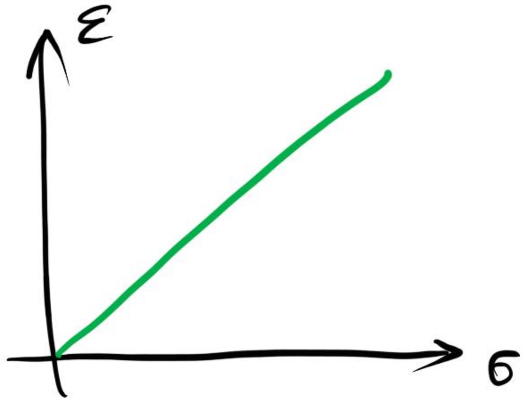
- Linear anisotropic strain-stress relationship

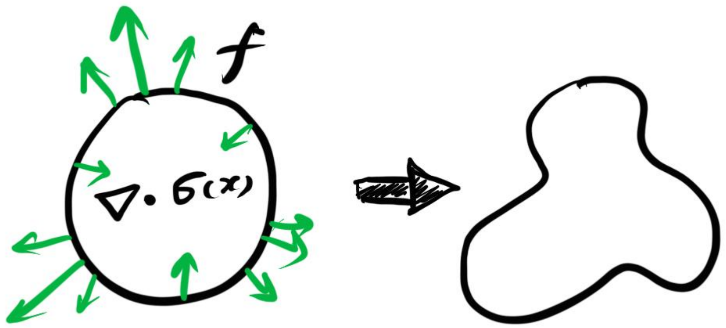
$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

21 independent variables in 3D

- Equation of elasticity

$$\boxed{C_{ijkl}\frac{\partial^2 u_k}{\partial x_l \partial x_j} + f_i = 0,}$$

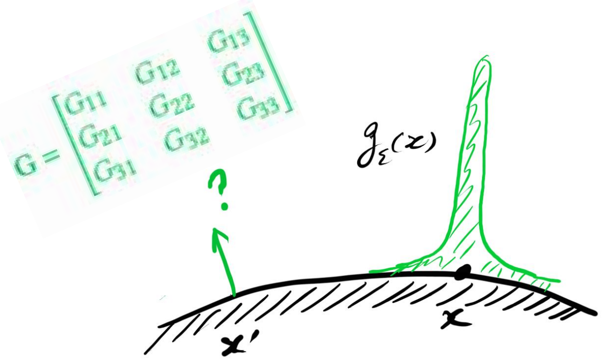




GREEN'S FUNCTION OF ELASTICITY

- **GF** to elasticity equation satisfies

$$\boxed{C_{ijkl}\frac{\partial^2 G_{km}}{\partial x_l \partial x_j} + \delta_{im} g_\varepsilon(\mathbf{x}) = 0,}$$



C_{ijkl} ⇒ Isotropic ⇒ $g_\varepsilon(\mathbf{x}) = \delta(\mathbf{x})$ ⇒ $u(r) = \left[\frac{(a-b)}{r} I + \frac{b}{r^3} r r^t\right] f$ on

[Kelvin 1848]

Isotropic ⇒ $g_\varepsilon(r) = 15\varepsilon^4/(8\pi)(r^2+\varepsilon^2)^{-\frac{7}{2}}$ ⇒ $u_\varepsilon(r) = \left[\frac{(a-b)}{r_\varepsilon} I + \frac{b}{r_\varepsilon^3} r r^t + \frac{a}{2}\frac{\varepsilon^2}{r_\varepsilon^3} I\right] f \equiv \mathcal{K}_\varepsilon(r) f.$

[Cortez 2001; de Goes 2017]

C_{ijkl} ⇒ Anisotropic ⇒ $g_\epsilon(x)$ is a more general radially symmetric function

???

GENERAL REGULARIZED GREEN'S FUNCTION

To derive **GF** from $\boxed{C_{ijkl}\dfrac{\partial^2 G_{km}}{\partial x_l \partial x_j} + \delta_{im} g_\varepsilon(\mathbf{x}) = 0,}$

- we use Fourier transform

$$\widehat{G}_{km}(\xi) = (C_{ijkl}\xi_l\xi_j)^{-1}\delta_{im}\boxed{\widehat{g}_\varepsilon(\xi).}$$

radial symmetry

$$\widehat{g}_\varepsilon(\xi) = \widehat{g}_\varepsilon(|\xi|) = \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{|\xi|}} \int_0^\infty J_{\frac{1}{2}}(|\mathbf{x}||\xi|) g_\varepsilon(|\mathbf{x}|)|\mathbf{x}|^{\frac{3}{2}}\, \mathrm{d}|\mathbf{x}|,$$

Spatial domain | Frequency domain

$f(x) \rightarrow \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$

$a \cdot f(x) \rightarrow a \cdot \hat{f}(\xi)$

$\dfrac{d^n f(x)}{dx^n} \rightarrow (i\xi)^n \hat{f}(\xi)$

- and its inverse Fourier transform

$$\boxed{\mathbf{G}(\mathbf{x}) = \frac{1}{8\pi^3}\int_{\mathbb{R}^3} \widehat{\mathbf{G}}(\xi) \exp(\mathbb{i}\,\mathbf{x}\cdot\xi)\, \mathrm{d}\xi,}$$

How to evaluate this integral?

- Thus, given a load function $f(x) = \delta(x - x')h$

$$\mathbf{u}(\mathbf{x}) = \mathrm{Re}[\mathbf{G}(\mathbf{x} - \mathbf{x}')]\,\mathbf{h},$$

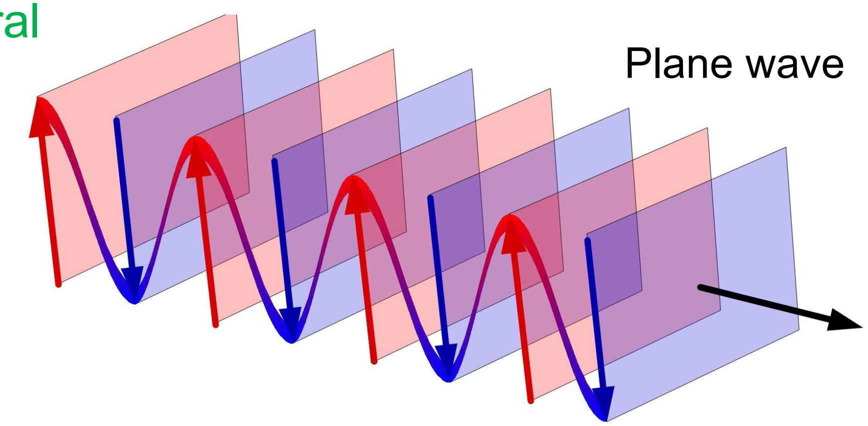
SPHERICAL HARMONIC EXPANSION

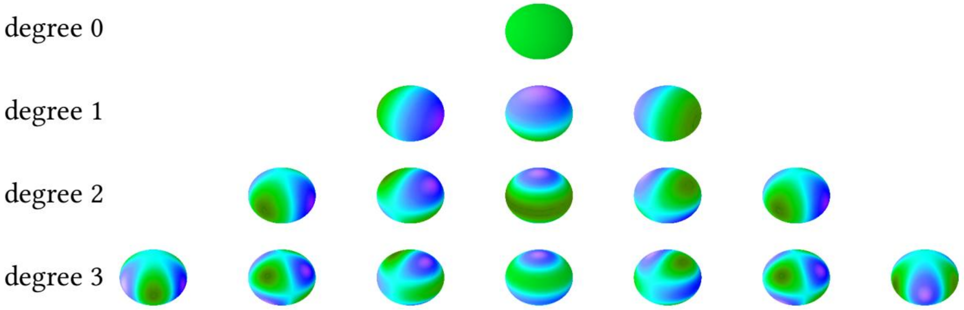
- For arbitrary material C_{ijkl}, $\mathbf{G}(\mathbf{x})$ has no analytical expressions in general
- Plane-wave expansion, or Rayleigh expansion

$$\exp(\mathbb{i}\,\mathbf{x}\cdot\xi) = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathbb{i}^l j_l(|\mathbf{x}||\xi|)\, Y_l^m(\widetilde{\mathbf{x}})\, \overline{Y}_l^m(\widetilde{\xi}),$$

- Recall: $\widehat{G}_{km}(\xi) = (C_{ijkl}\xi_l\xi_j)^{-1}\delta_{im}\widehat{g}_\varepsilon(\xi)$.

$$\mathbf{G}(\mathbf{x}) = \frac{1}{2\pi^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathbb{i}^l\, Y_l^m(\widetilde{\mathbf{x}}) \int_0^{\infty} \widehat{g}_\varepsilon(|\xi|)\, j_l(|\mathbf{x}||\xi|)\, \mathrm{d}|\xi| \cdot \int_{\mathbb{S}^2} (C_{ikjl}\widetilde{\xi}_k\widetilde{\xi}_l)^{-1}\, \overline{Y}_l^m(\widetilde{\xi})\, \mathrm{d}\mathbb{S}(\widetilde{\xi}),$$


Plane wave



Spherical harmonics

GREEN'S FUNCTION IN SERIES

- Expressed in spherical coordinates, $\mathbf{G}(\mathbf{x})$ is decomposed as

Directional term

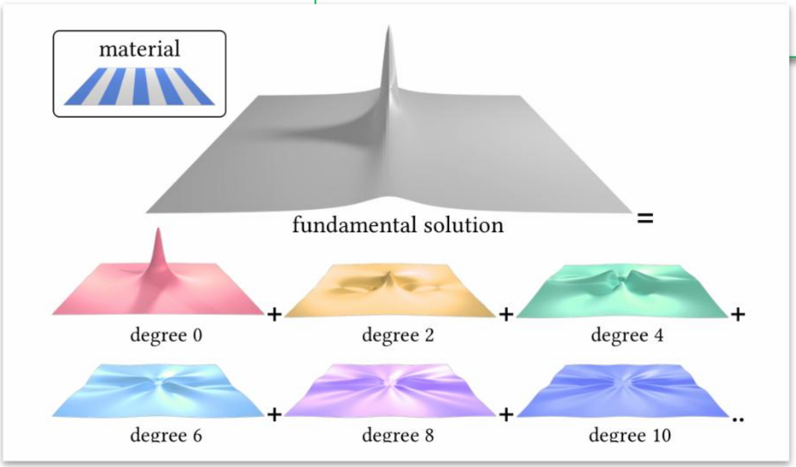
Radial term

$$\int_0^{\infty} \widehat{g}_{\varepsilon}(|\xi|)\, j_l(|\mathbf{x}||\xi|)\, \mathrm{d}|\xi|$$

$$\mathbf{G}(r,\theta,\varphi) = \frac{1}{2\pi^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{i}^l\, Y_l^m(\theta,\varphi)\, R_l(r)\, \mathbf{P}_l^m(\mathbf{C}),$$

$$\int_{\mathbb{S}^2} (C_{ikjl}\tilde{\xi}_k\tilde{\xi}_l)^{-1}\, \overline{Y_l^m}(\tilde{\xi})\, \mathrm{d}\mathbb{S}(\tilde{\xi})$$

Material term



ISOTROPIC CASE: KELVINLETS [DEGOES & JAMES 2017]

- For isotropic material, $C_{ijkl} \Rightarrow \mu, \lambda$
- $\mathbf{G}(\mathbf{x})$ are only non-zero at degree **0** and **2**.

$$g_\varepsilon(r) = \delta(r)$$

$$\begin{aligned}
G_{11}(r,\theta,\varphi) &= \frac{\sin^2(\theta)(\lambda+\mu)\cos^2(\varphi)+\lambda+3\mu}{8\pi\mu r(\lambda+2\mu)},\\
G_{12}(r,\theta,\varphi) = G_{21}(r,\theta,\varphi) &= \frac{\sin^2(\theta)(\lambda+\mu)\sin(\varphi)\cos(\varphi)}{8\pi\mu r(\lambda+2\mu)},\\
G_{13}(r,\theta,\varphi) = G_{31}(r,\theta,\varphi) &= \frac{\sin(\theta)\cos(\theta)(\lambda+\mu)\cos(\varphi)}{8\pi\mu r(\lambda+2\mu)},\\
G_{22}(r,\theta,\varphi) &= \frac{\sin^2(\theta)(\lambda+\mu)\sin^2(\varphi)+\lambda+3\mu}{8\pi\mu r(\lambda+2\mu)},\\
G_{23}(r,\theta,\varphi) = G_{32}(r,\theta,\varphi) &= \frac{\sin(\theta)\cos(\theta)(\lambda+\mu)\sin(\varphi)}{8\pi\mu r(\lambda+2\mu)},\\
G_{33}(r,\theta,\varphi) &= \frac{\lambda\cos^2(\theta)+\mu\cos^2(\theta)+\lambda+3\mu}{8\pi\mu r(\lambda+2\mu)}
\end{aligned}$$

$$\boldsymbol{u}(\boldsymbol{r}) = \boxed{\left[\frac{(a-b)}{r}\boldsymbol{I} + \frac{b}{r^3}\boldsymbol{r}\boldsymbol{r}^t\right]}\boldsymbol{f}$$

Equivalent to Kelvinlet

$$g_\varepsilon(r) = 15\varepsilon^4/(8\pi)(r^2+\varepsilon^2)^{-\frac{7}{2}}$$

$$\begin{aligned}
G_{11}(r,\theta,\varphi) &= \frac{r^2\sin^2(\theta)(\lambda+\mu)\cos^2(\varphi)+r^2(\lambda+3\mu)+\varepsilon^2(2\lambda+5\mu)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},\\
G_{12}(r,\theta,\varphi) = G_{21}(r,\theta,\varphi) &= \frac{r^2\sin^2(\theta)(\lambda+\mu)\sin(\varphi)\cos(\varphi)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},\\
G_{13}(r,\theta,\varphi) = G_{31}(r,\theta,\varphi) &= \frac{r^2\sin(\theta)\cos(\theta)(\lambda+\mu)\cos(\varphi)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},\\
G_{22}(r,\theta,\varphi) &= \frac{r^2\sin^2(\theta)(\lambda+\mu)\sin^2(\varphi)+r^2(\lambda+3\mu)+\varepsilon^2(2\lambda+5\mu)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},\\
G_{23}(r,\theta,\varphi) = G_{32}(r,\theta,\varphi) &= \frac{r^2\sin(\theta)\cos(\theta)(\lambda+\mu)\sin(\varphi)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}},\\
G_{33}(r,\theta,\varphi) &= \frac{r^2\cos^2(\theta)(\lambda+\mu)+r^2(\lambda+3\mu)+\varepsilon^2(2\lambda+5\mu)}{8\pi\mu(\lambda+2\mu)\left(r^2+\varepsilon^2\right)^{3/2}}.
\end{aligned}$$

$$\boldsymbol{u}_\varepsilon(\boldsymbol{r}) = \boxed{\left[\frac{(a-b)}{r_\varepsilon}\boldsymbol{I} + \frac{b}{r_\varepsilon^3}\boldsymbol{r}\boldsymbol{r}^t + \frac{a}{2}\frac{\varepsilon^2}{r_\varepsilon^3}\boldsymbol{I}\right]}\boldsymbol{f}$$

EXTENSION TO GRADIENT

- Fourier transform of partial derivative

$$\widehat{G_{ij,p}} = \mathbb{i}\,\xi_p\,\widehat{G}_{ij}, \quad p = 0, 1, 2,$$

$$\mathcal{R}_l(|\mathbf{x}|) = \int_0^\infty \widehat{g}_\varepsilon(|\xi|)\, j_l(|\mathbf{x}||\xi|)\, |\xi| \,\mathrm{d}|\xi|,$$

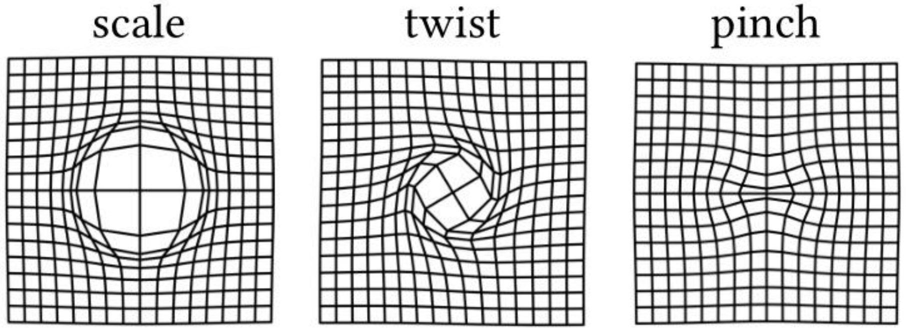
- SH expansion of gradient

$$\mathcal{P}^m_{l,p}(\mathrm{C}) = \int_{\mathbb{S}^2} (C_{ikjl}\tilde{\xi}_k\tilde{\xi}_l)^{-1}\,\overline{Y_l^m}(\tilde{\xi})\,\tilde{\xi}_p \,\mathrm{d}\mathbb{S}(\tilde{\xi}).$$

$$\boxed{\nabla_p \mathrm{G}(\mathbf{x}) = \frac{1}{2\pi^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathbb{i}^{l+1}\, Y_l^m(\tilde{\mathbf{x}})\, \mathcal{R}_l(|\mathbf{x}|)\, \mathcal{P}^m_{l,p}(\mathrm{C})}$$

- Then given an affine load $F(x) = \delta(x - x')H$

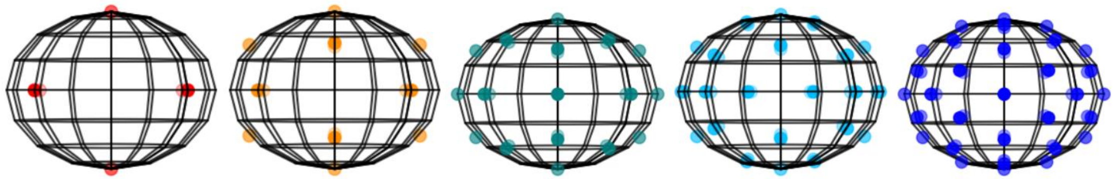
$$\mathbf{u}(\mathbf{x}) = \mathrm{Re}\left[\nabla\mathbf{G}(\mathbf{x} - \mathbf{x}')\right] : \mathbf{H},$$



IMPLEMENTATION

Offline computations

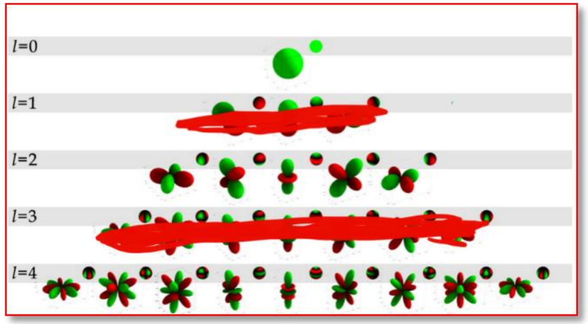
- $\mathbf{P}_l^m(\mathbf{C}) = \int_{\mathbb{S}^2} (C_{ikjl}\tilde{\xi}_k\tilde{\xi}_l)^{-1}\overline{Y}_l^m(\tilde{\xi})\,\mathrm{d}\mathbb{S}(\tilde{\xi})$
- and $\mathcal{P}_{l,p}^m(\mathrm{C}) = \int_{\mathbb{S}^2} (C_{ikjl}\tilde{\xi}_k\tilde{\xi}_l)^{-1}\overline{Y}_l^m(\tilde{\xi})\,\tilde{\xi}_p\,\mathrm{d}\mathbb{S}(\tilde{\xi}).$
- No analytical expressions in general
- Can be precomputed using *Lebedev* quadrature



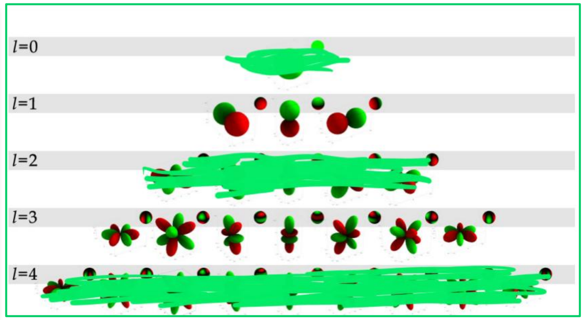
order = 6 order = 14 order = 26 order = 38 order = 50

Online computations

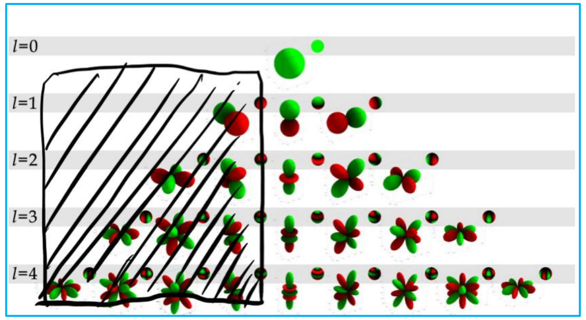
- $\mathbf{G}_l(\mathbf{x}) \equiv 0$ for $l = 1,3,5\ldots$



- $\nabla\mathbf{G}_l(\mathbf{x}) \equiv 0$ for $l = 0,2,4\ldots$



- $\mathrm{Re}[\mathbf{G}_l^{-m}(\mathbf{x})] \equiv \mathrm{Re}[\mathbf{G}_l^{m}(\mathbf{x})]$
- $\mathrm{Re}[\nabla\mathbf{G}_l^{-m}(\mathbf{x})] \equiv \mathrm{Re}[\nabla\mathbf{G}_l^{m}(\mathbf{x})]$
- SH series truncation
- Parallelization



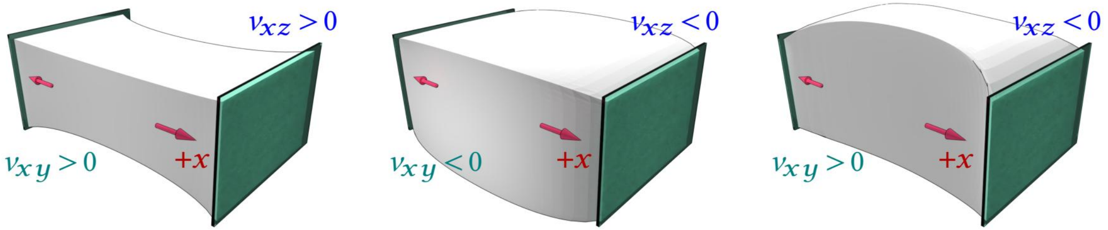
EVALUATIONS & RESULTS

ANISOTROPY CONTROL

PARTICULAR TYPES OF MATERIAL

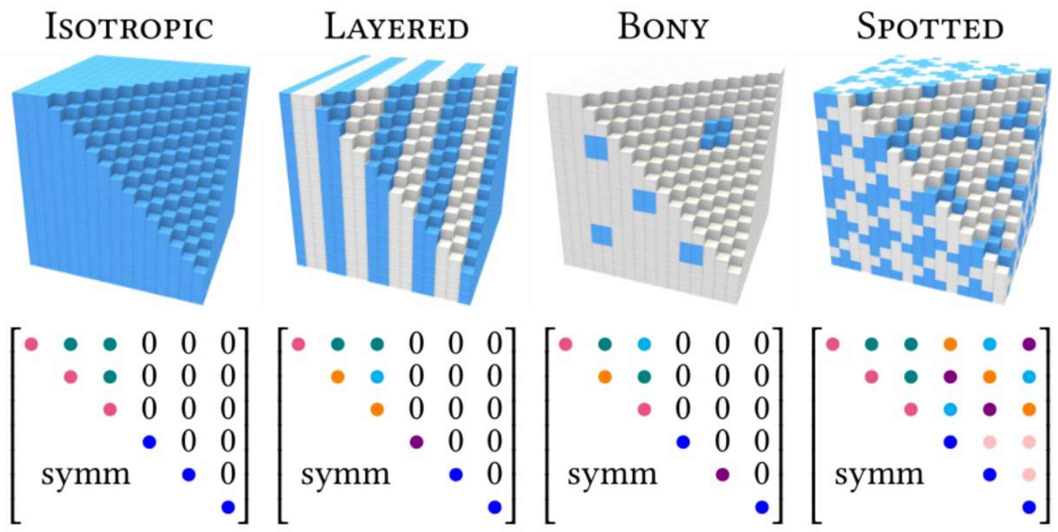
- e.g., orthotropic material with 3 Young's moduli, 3 shear moduli and 3 Poisson ratios

$$
\mathbf{S}^V_{\text{orth}} = \begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\
-\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\
-\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{zx}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}}
\end{bmatrix},
$$



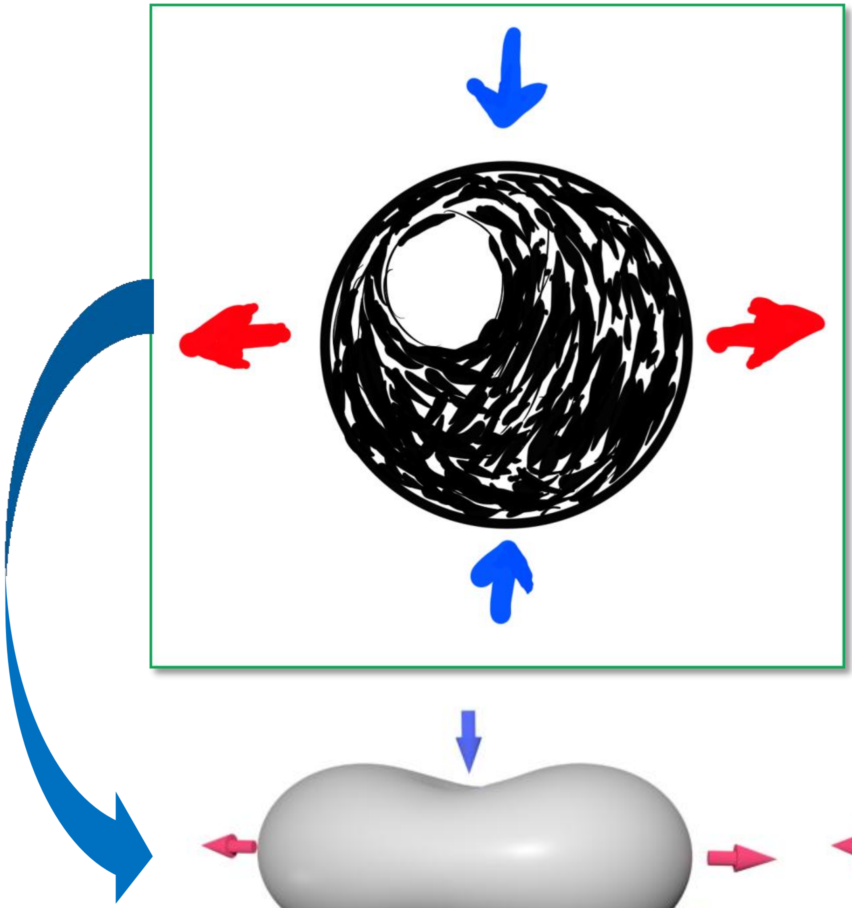
GENERAL CASES

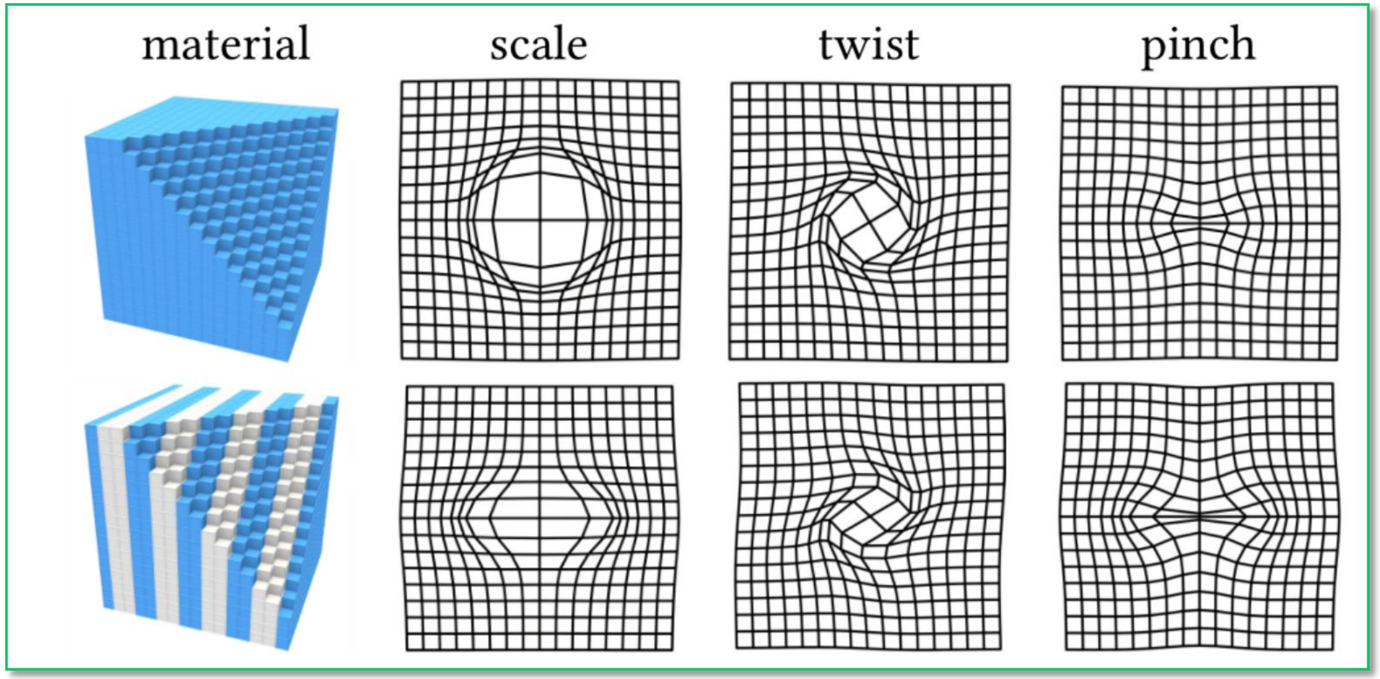
- Homogenize bi-materials on a regular grid [Kharevych et al. 2008]

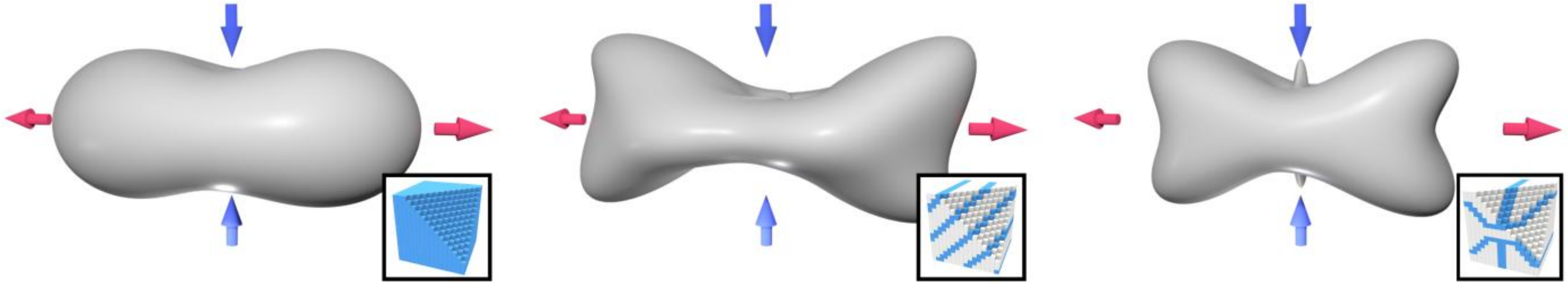


C_{ijkl} in Voigt notation

ANISOTROPY CONTROL



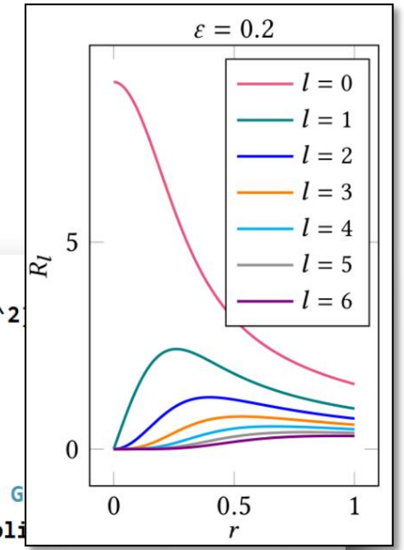




DEFORMATION PROPAGATION CONTROL

SIGGRAPH 2022 VANCOUVER+ 8-11 AUG

- Recall: $R_l(|x|) = \int_0^\infty \widehat{g}_\varepsilon(|\xi|)\, j_l(|\mathbf{x}||\xi|)\, \mathrm{d}|\xi|$ which controls deformation propagation
- For specific smooth impulses..




```
In[3]:= (* smooth load [de Goes and James 2017] *)
g[r_] := 15 ϵ^4 / (8 Pi) Power[r^2 + ϵ^2, -7 / 2]
```

Out[4]= $\dfrac{15\,\epsilon^4}{8\,\pi\,(r^2+\epsilon^2)^{7/2}}$

$g_\epsilon(r)$

```
(* Fourier transform of g(r): for regularized Green's function *)
hatg = Power[2 Pi, 3 / 2] / Sqrt[ξ] Integrate[Simplify[BesselJ[1 / 2, r ξ]
    g[r] Power[r, 3 / 2]],
   {r, 0, Infinity}, Assumptions → ξ ≥ 0 && ϵ > 0]
```

Out[5]= $\frac{1}{2}\,\epsilon^2\,\xi^2\,\mathrm{BesselK}[2, \epsilon\,\xi]$

$\widehat{g}_\epsilon(r)$

```
(* compute the radial term R_l *)
R = Integrate[hatg * SphericalBesselJ[l, r ξ], {ξ, 0, Infinity},
  Assumptions → ϵ > 0 && l ≥ 0 && r > 0]
```

$R_l(r)$

Out[6]= $\frac{1}{2}\sqrt{\pi}\, r^l\, \epsilon^{-1-l}\, \mathrm{Gamma}\left[\frac{1+l}{2}\right] \mathrm{Gamma}\left[\frac{5+l}{2}\right] \mathrm{Hypergeometric2F1Regularized}\left[\frac{1+l}{2}, \frac{5+l}{2}, \frac{3}{2}+l, -\frac{r^2}{\epsilon^2}\right]$

```
(* Gaussian load *)
g[r_] := 1 / (ϵ^3 Power[Pi, 3 / 2]) Exp[-r^2 / ϵ^2]
```

Out[14]= $\dfrac{e^{-\frac{r^2}{\epsilon^2}}}{\pi^{3/2}\,\epsilon^3}$

```
In[15]:= (* Fourier transform of g(r): for regularized G
hatg = Power[2 Pi, 3 / 2] / Sqrt[ξ] Integrate[Simpli
    g[r] Power[r, 3 / 2]],
   {r, 0, Infinity}, Assumptions → ξ ≥ 0 && ϵ > 0]
```

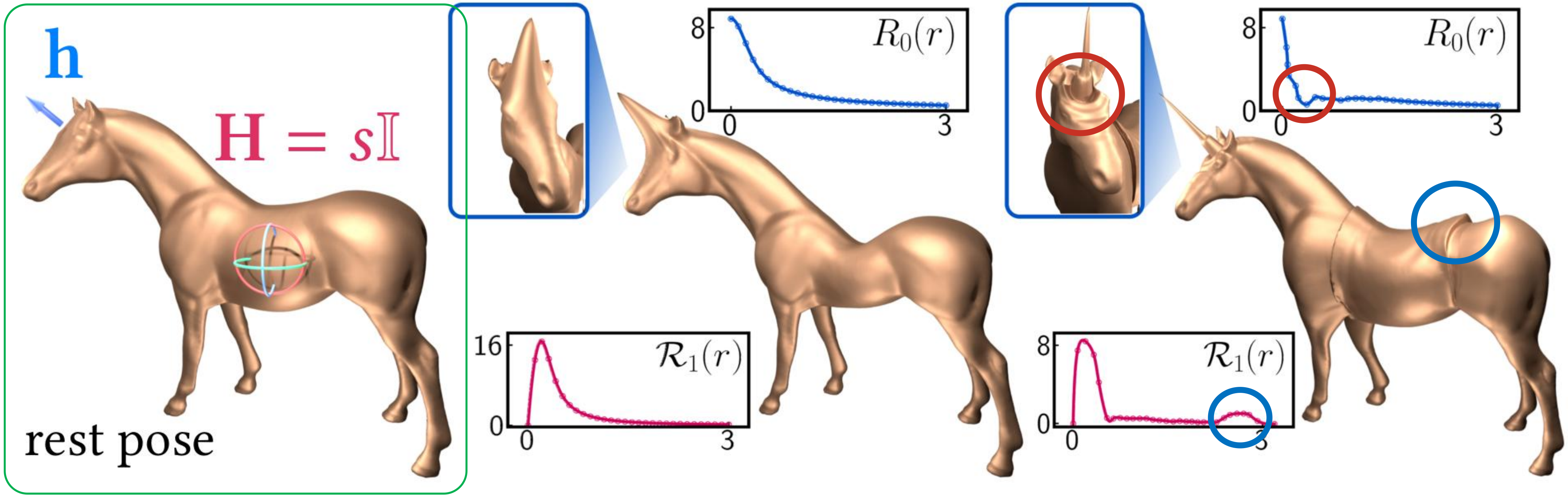
Out[15]= $e^{-\frac{1}{4}\epsilon^2\xi^2}$

```
In[16]:= (* compute the radial term R_l *)
R = Integrate[hatg * SphericalBesselJ[l, r ξ], {ξ, 0, Infinity},
  Assumptions → ϵ > 0 && l ≥ 0 && r > 0]
```

Out[16]= $\frac{1}{2}\sqrt{\pi}\, r^l\, \epsilon^{-1-l}\, \mathrm{Gamma}\left[\frac{1+l}{2}\right] \mathrm{Hypergeometric1F1Regularized}\left[\frac{1+l}{2}, \frac{3}{2}+l, -\frac{r^2}{\epsilon^2}\right]$

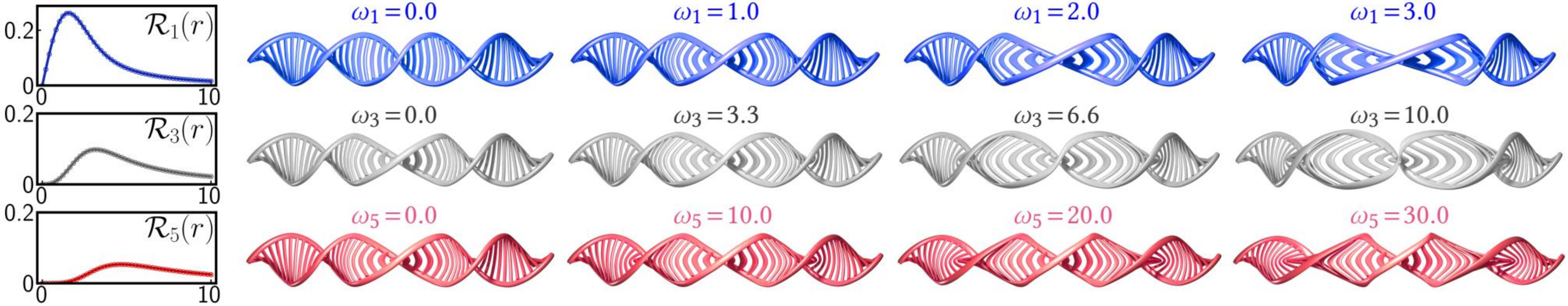
DEFORMATION PROPAGATION CONTROL

- **Limitations:**
  - $g_\epsilon(r)$ is neither intuitive nor flexible
  - the integral $R_l(r)$ (or $\mathcal{R}_l(r)$) is hard to evaluate given $g_\epsilon(r)$
    - may not even exist!
- **Our approach**:
  - Edit $R_l(r)$ (or $\mathcal{R}_l(r)$) directly via cubic splines instead of constructing an integrable $g_\epsilon(r)$



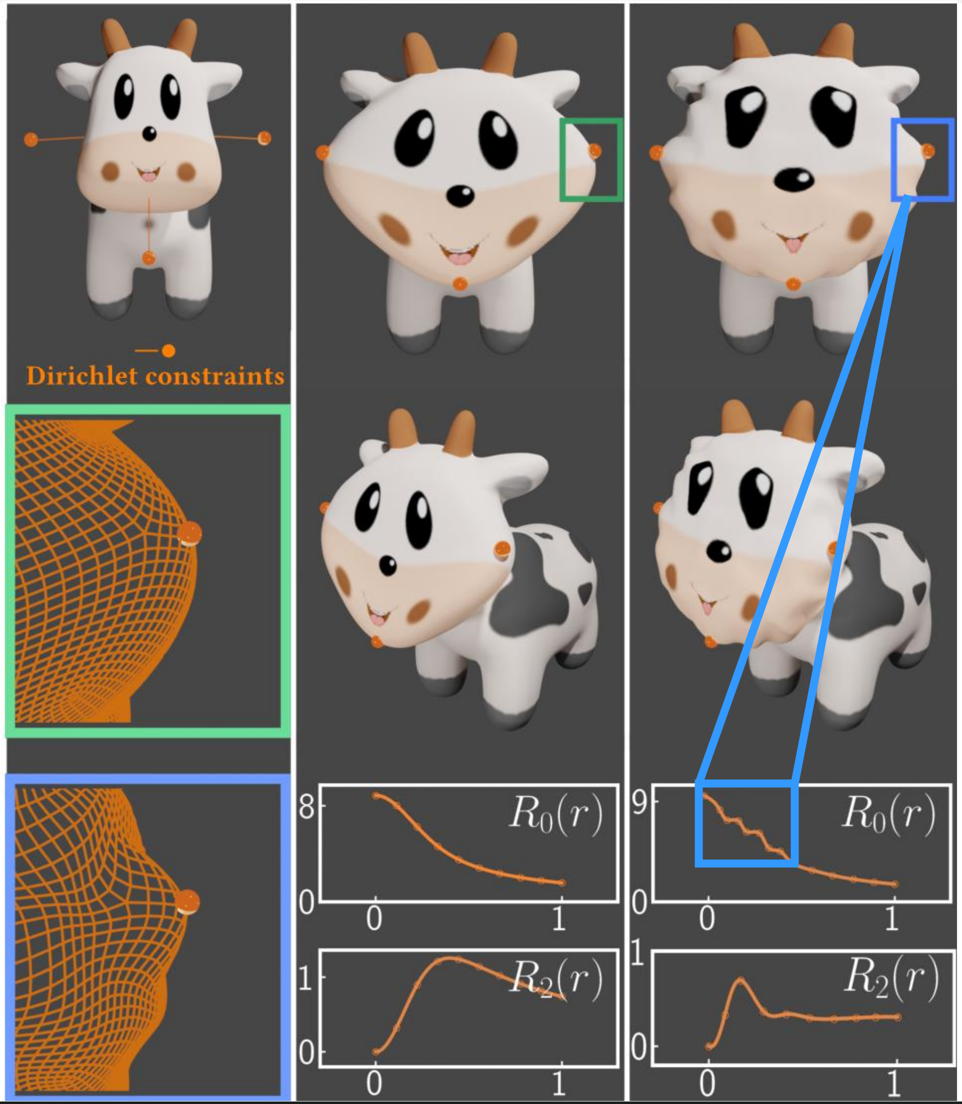
DEFORMATION PROPAGATION CONTROL

- Deformation propagation of each degree



Small scaling → Big scaling

CONSTRAINED DEFORMATION



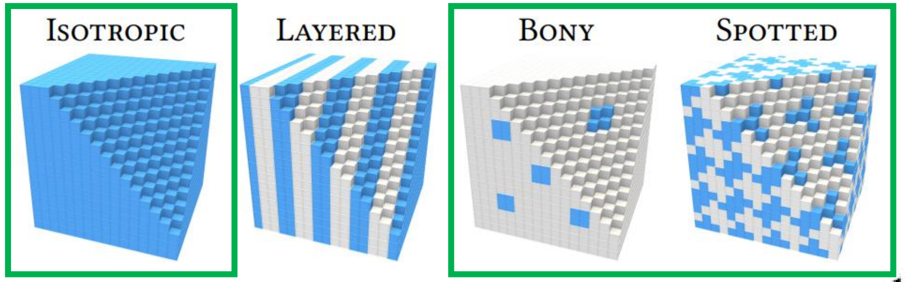
- Solve a dense linear system

$$\begin{bmatrix} \mathrm{Re}[\mathbf{G}(\mathbf{x}_0 - \mathbf{x}_0)] & \cdots & \mathrm{Re}[\mathbf{G}(\mathbf{x}_0 - \mathbf{x}_{k-1})] \\ \vdots & \ddots & \vdots \\ \mathrm{Re}[\mathbf{G}(\mathbf{x}_{k-1} - \mathbf{x}_0)] & \cdots & \mathrm{Re}[\mathbf{G}(\mathbf{x}_{k-1} - \mathbf{x}_{k-1})] \end{bmatrix} \begin{bmatrix} \mathbf{h}_0 \\ \vdots \\ \mathbf{h}_{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_{k-1} \end{bmatrix}$$

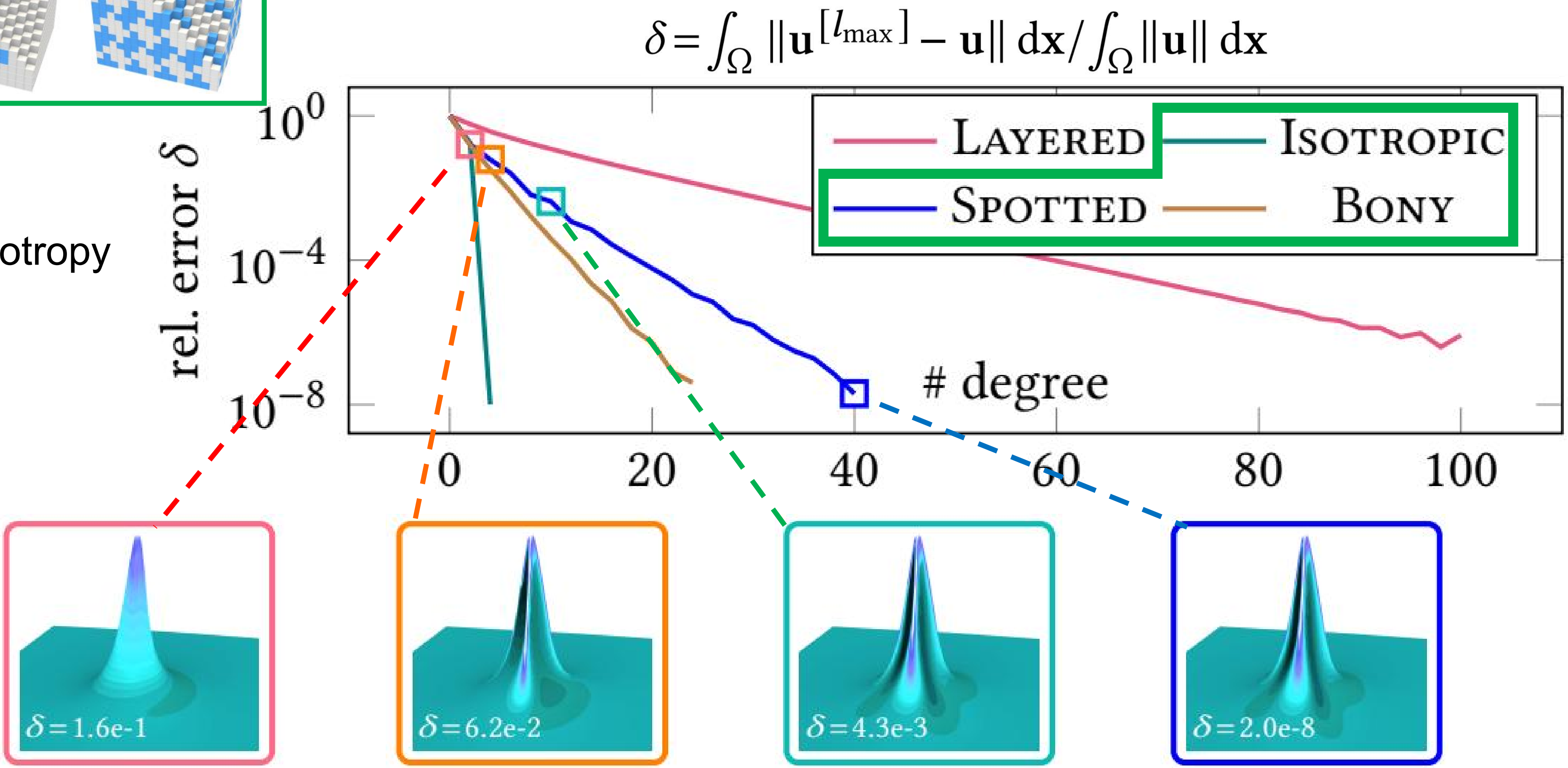
- By superposition:

$$\boxed{\mathbf{u}(\mathbf{x}) = \sum_{i=0}^{k} \mathbf{G}(\mathbf{x} - \mathbf{x}_i)\mathbf{h}_i}$$

TRUNCATING SPHERICAL HARMONIC SERIES



- $l_{\max} = 20$ for strong anisotropy
- $l_{\max} = 8$, otherwise

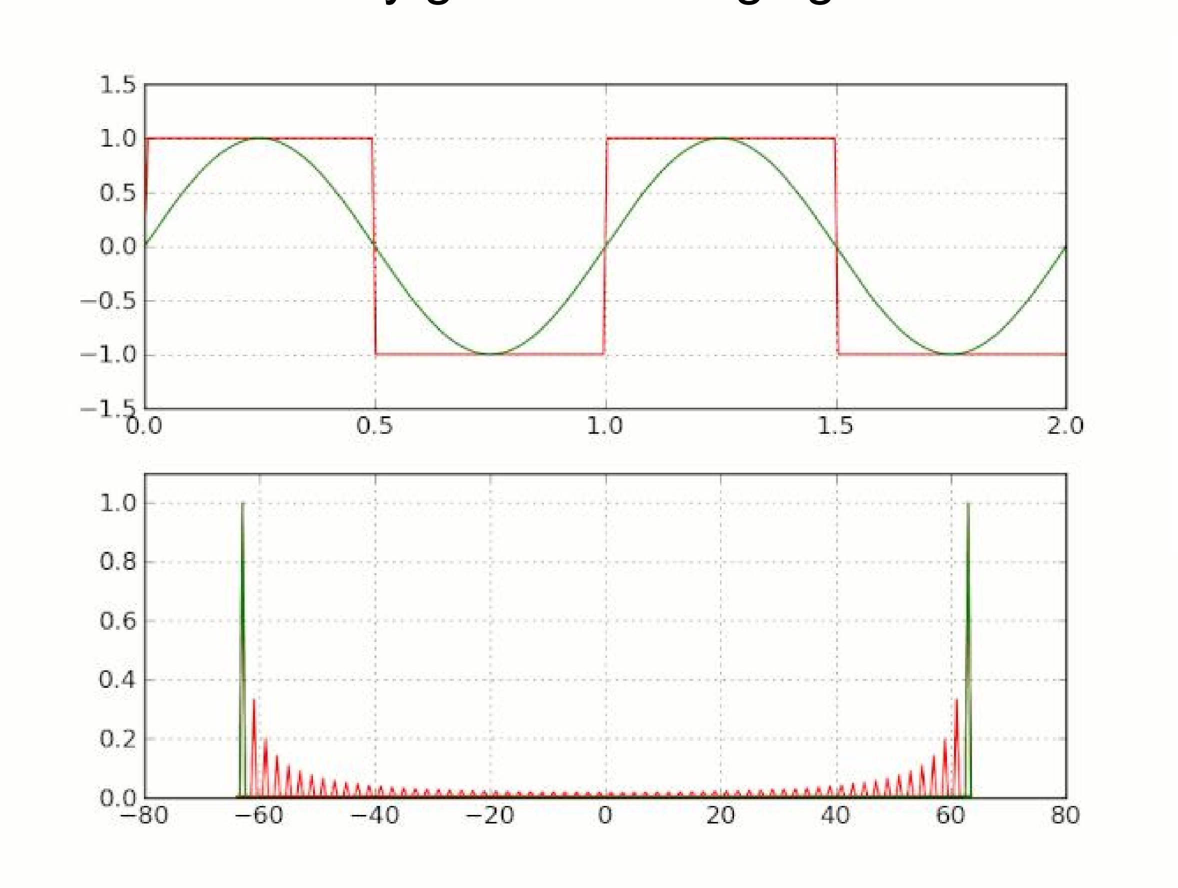


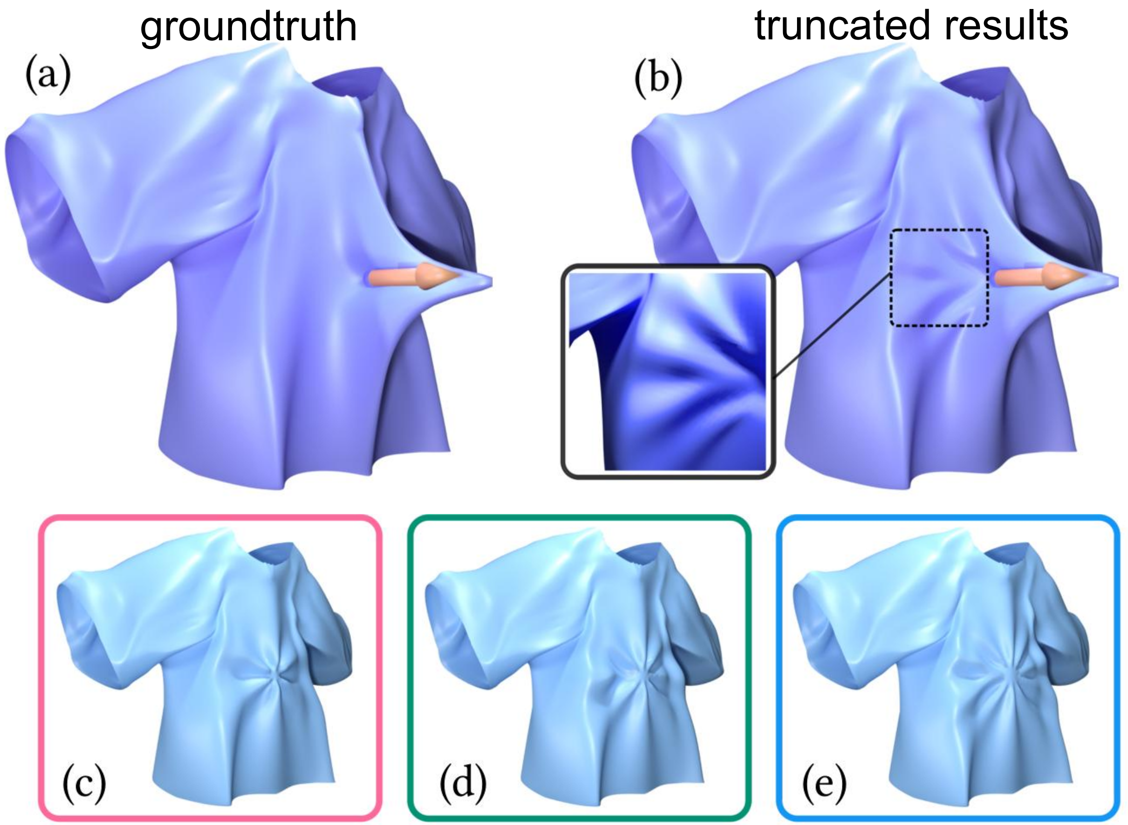
EXPLOIT EARLY TRUNCATION

Turn it into a feature!

Gibbs phenomenon:
discontinuity generates ringing artifacts!

Wrinkle generation

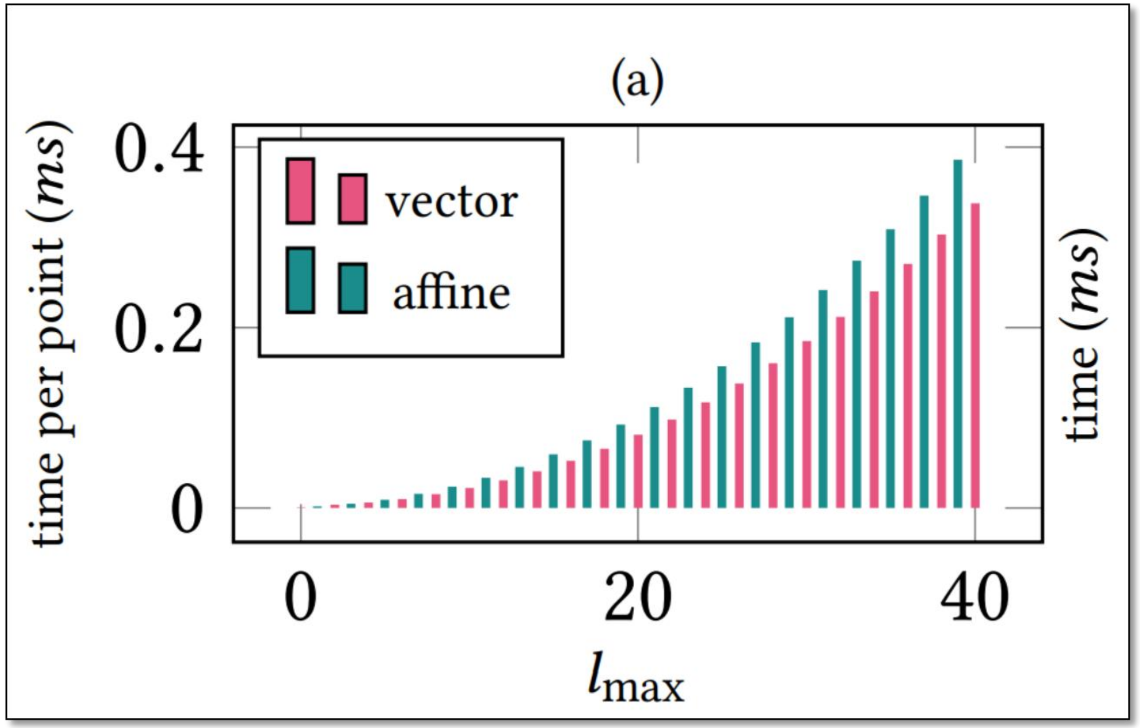




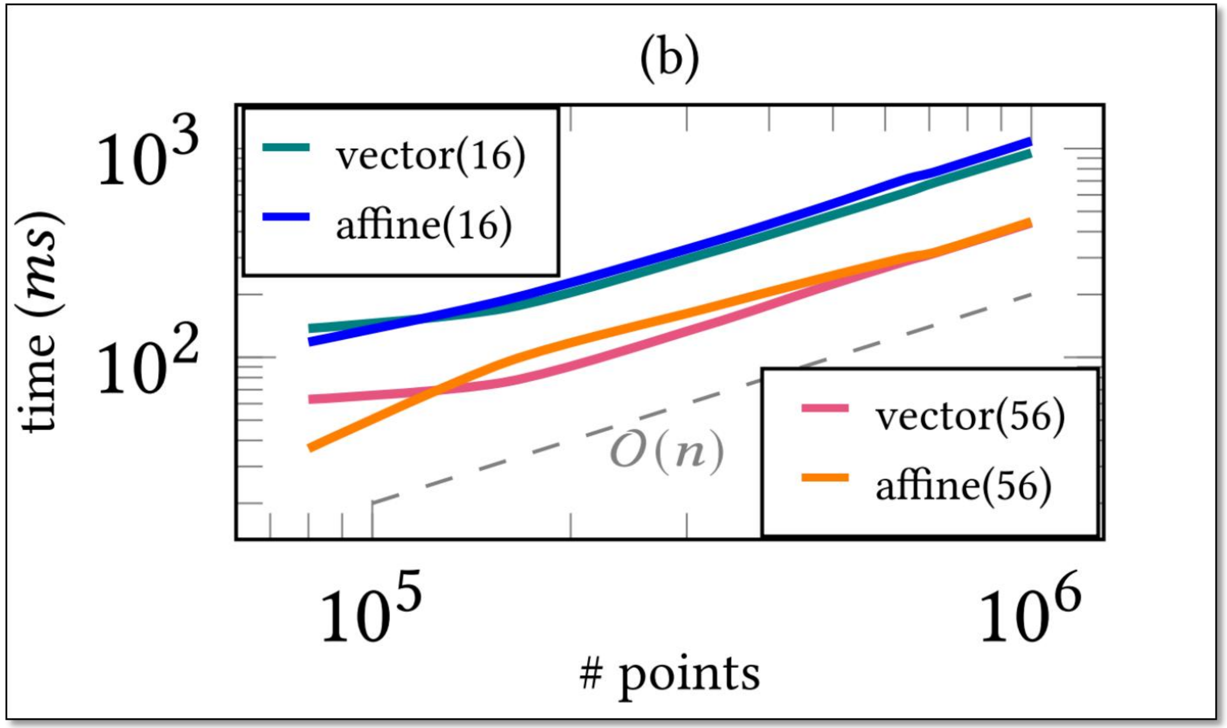
degree 6 degree 8 degree 10

TIMING

- Time complexity of pointwise evaluation

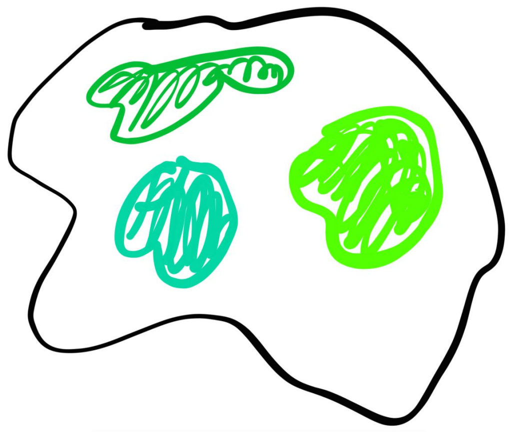


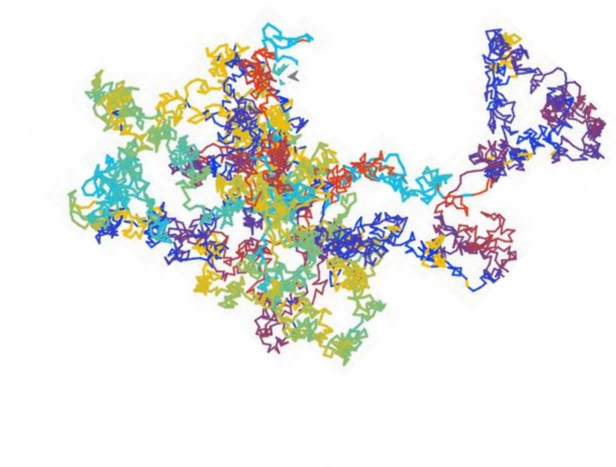
- Time complexity in problem size



FUTURE WORK

- Extend to PDEs with inhomogeneous coefficients
  - Convolution in the frequency domain
- Extend to elastodynamics
  - ODE solves in frequency domain
- Efficient handling of boundary conditions
  - Solve dense equations, fast multipole method
- Stochastic approach by random walks
  - Model probability distribution via SH
  - “Walk-on-bumpy-Spheres”





THANKS!