Numerical Coarsening using Discontinuous Shape Functions

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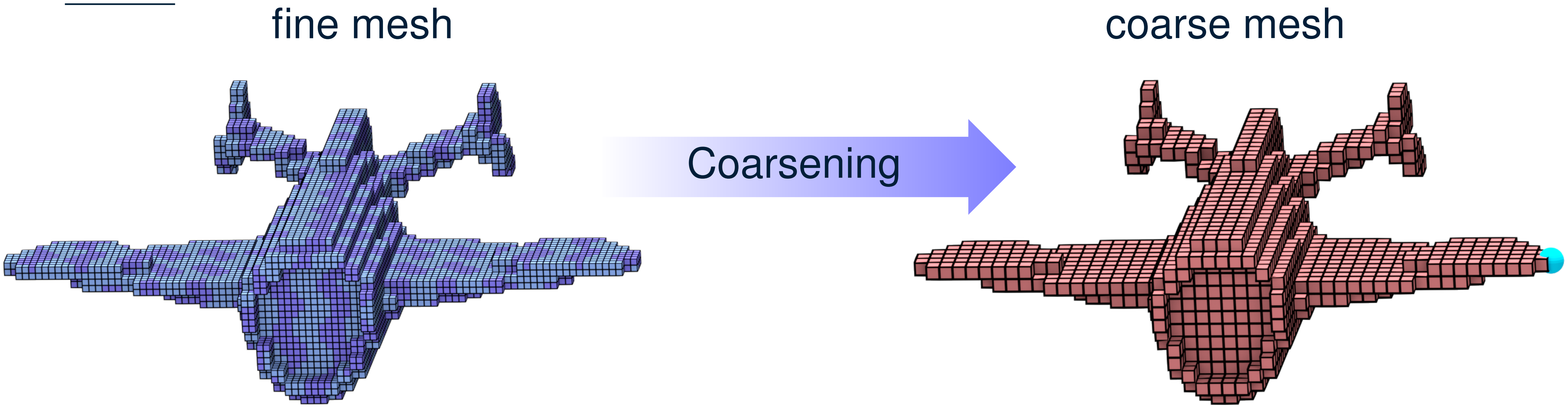
Photography & Recording Encouraged

Challenge

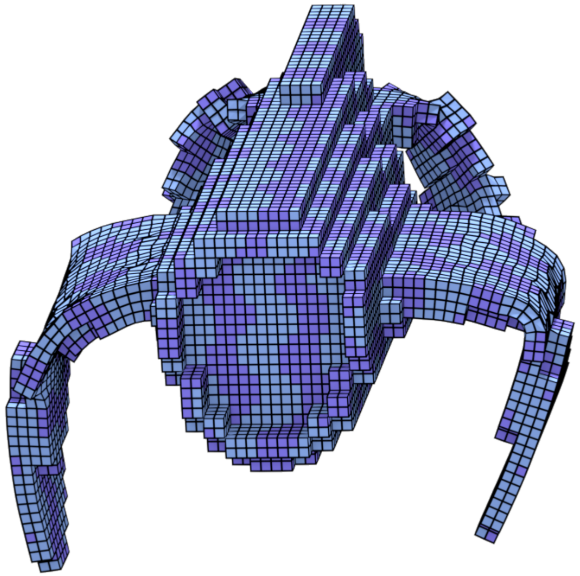


inhomogeneous & nonlinear material

Challenge

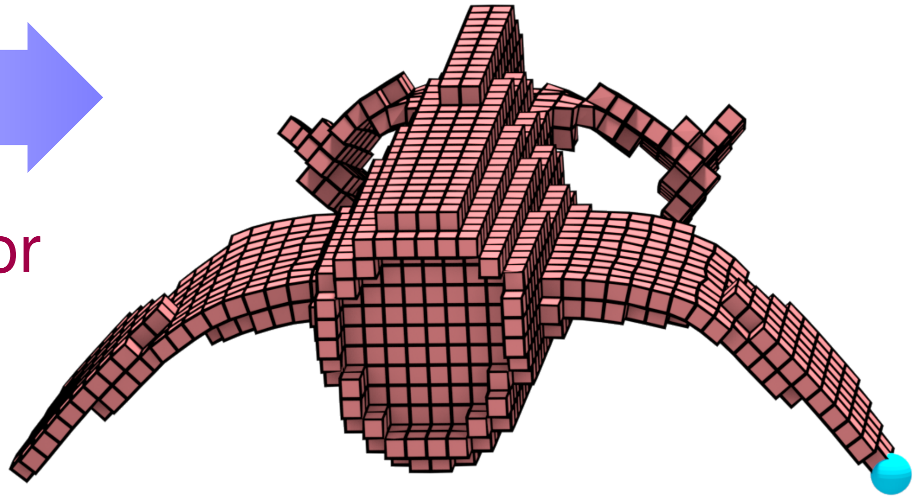
fine mesh

coarse mesh



Coarsening

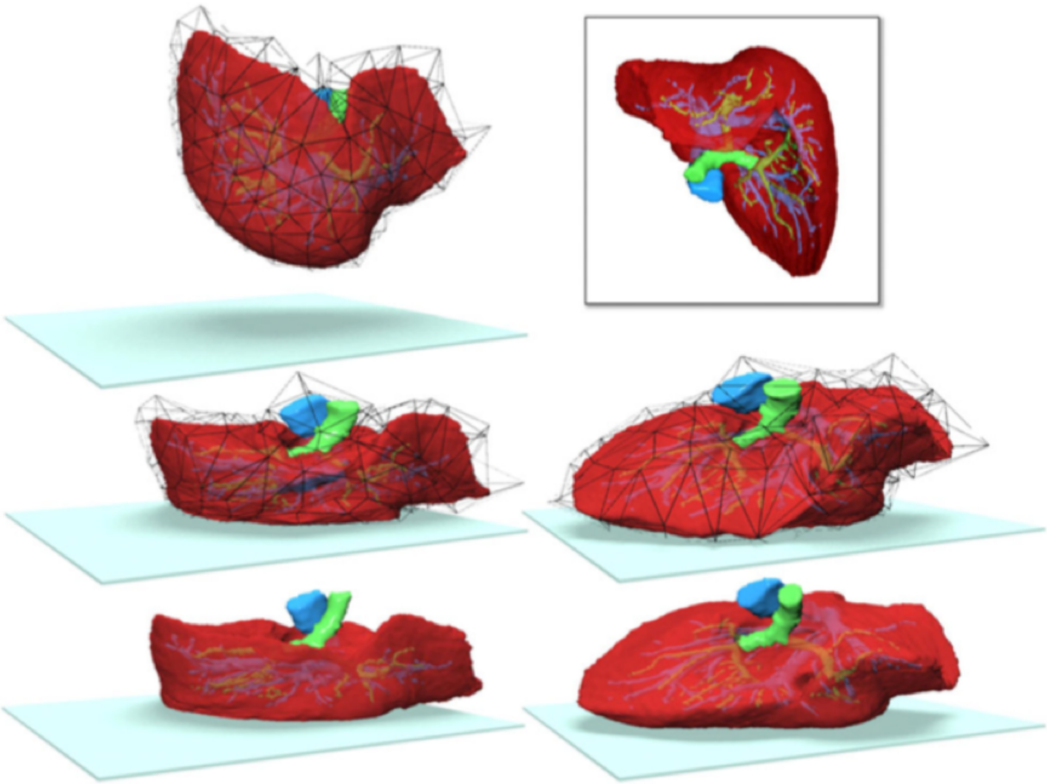
For similar behavior



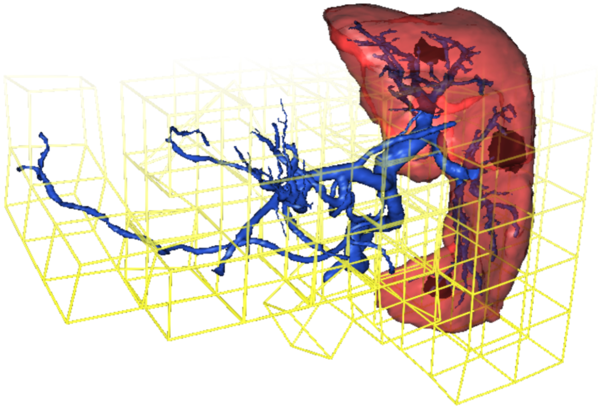
inhomogeneous & nonlinear material

what is the coarsened counterpart?

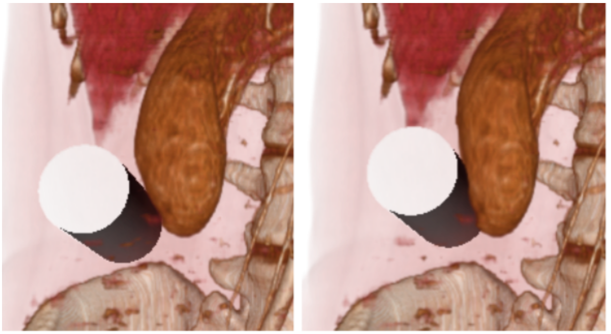
Previous Works



[Kharevych 2009]



[Nesme 2009]



[Torres 2016]

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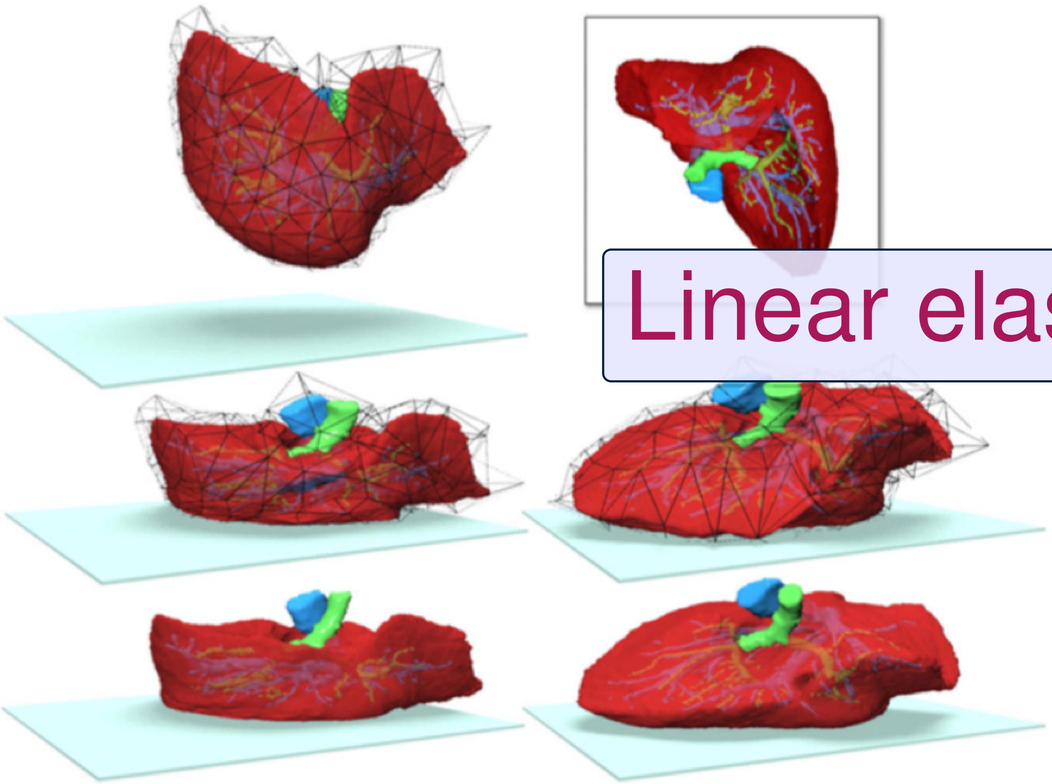
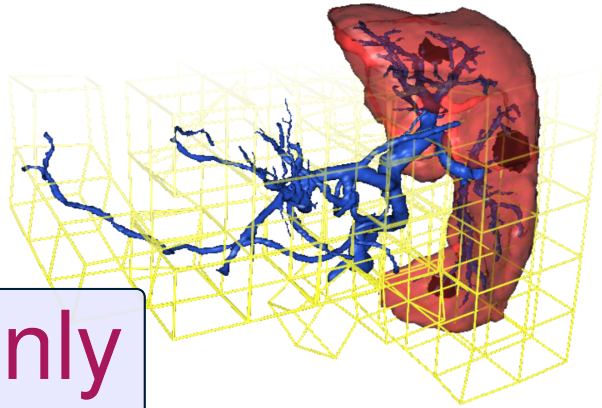


[Kharevych 2009]

Linear elasticity only



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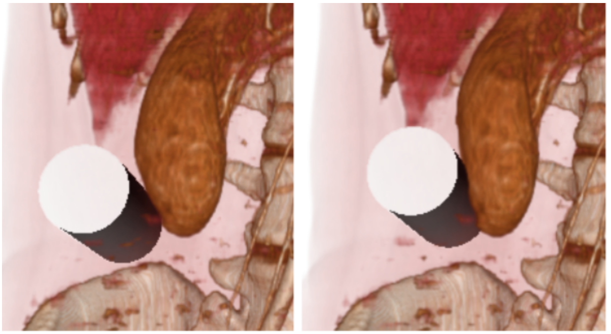

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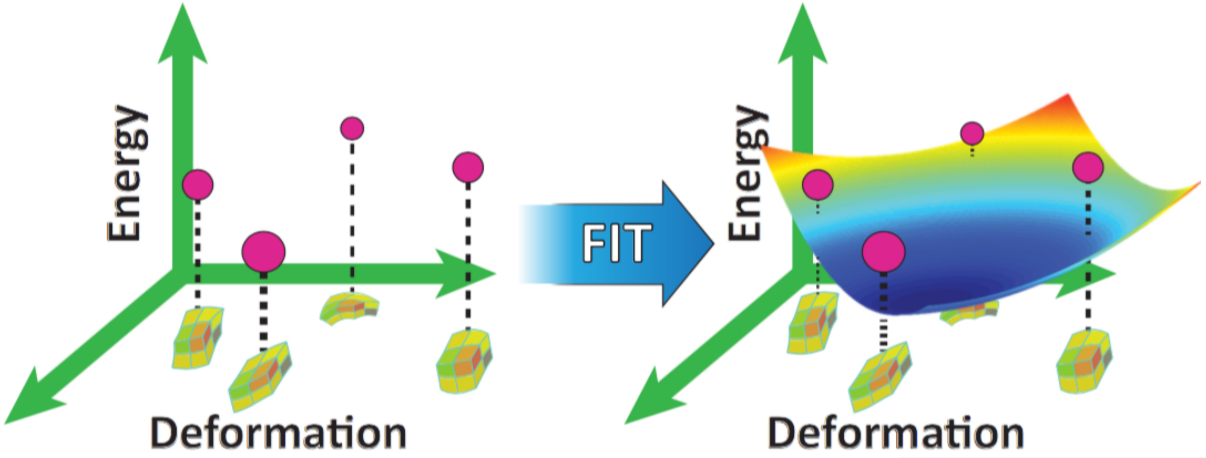
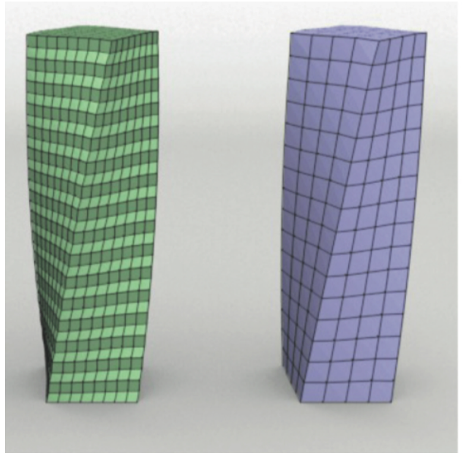
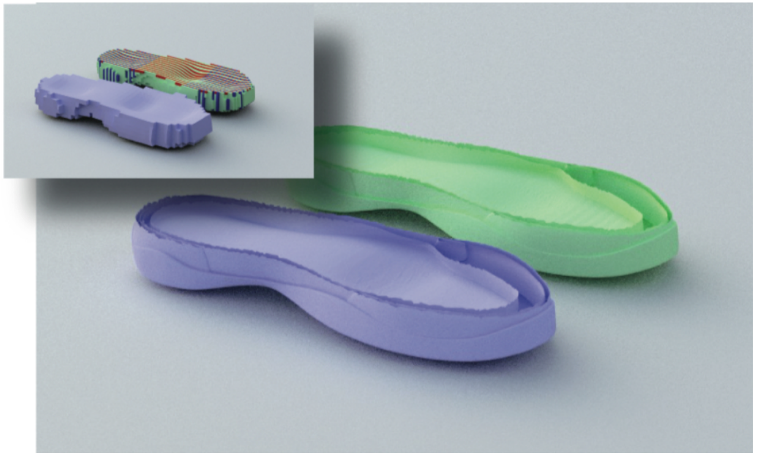


Data-driven (regression)

[Chen 2015]





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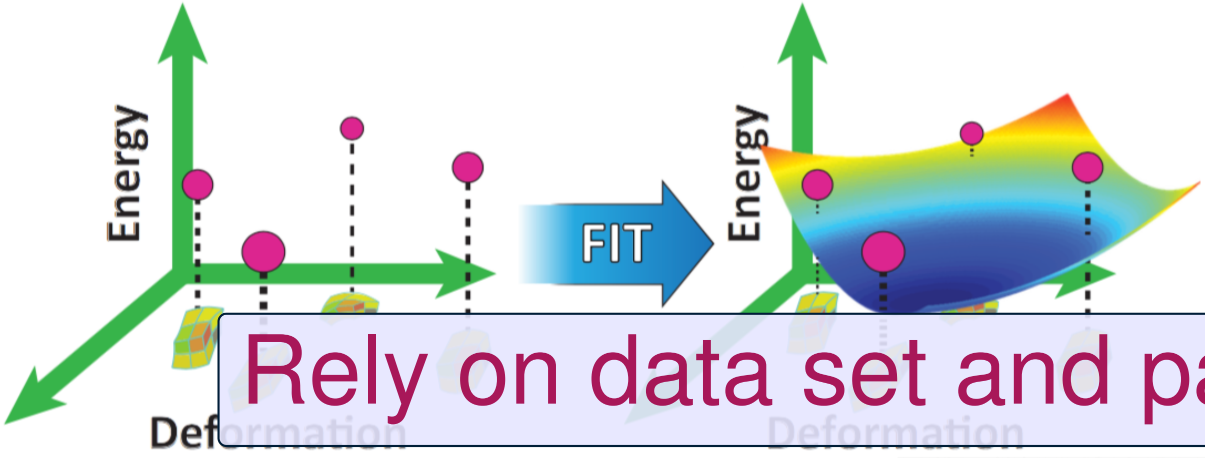
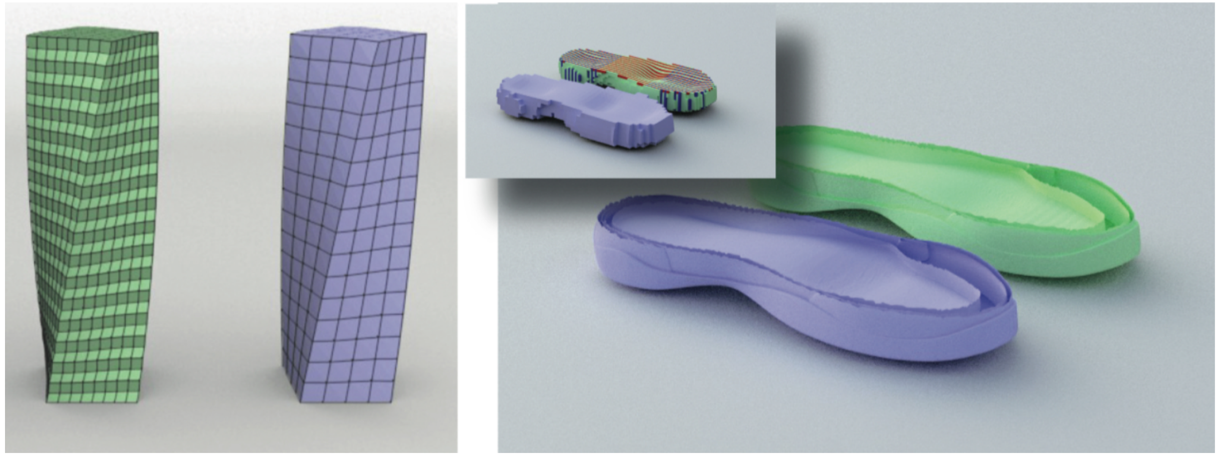


Data-driven (regression)

Rely on data set and parameter tuning

[Chen 2015]



Our Approach

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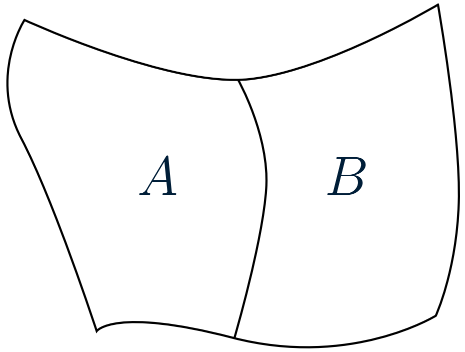
- NOT Homogenize the constitutive model
  - which template energy model would you use anyway?

Our Approach

- NOT Homogenize the constitutive model
  - which template energy model would you use anyway?
- BUT Approximate the solution space better
  - we can “adapt” the shape functions to the problem
  - estimate the shape on the fine mesh via the adapted shape functions
  - reuse input material via quadrature evaluations at runtime

The Implicit Constraints

- Solving deformation via variational formulation $\min_u \Psi[u(p)]$.
- continuous
  - in all possible functions
    $\forall u(p)$
  - full boundary conforming
    $u_A(p) = u_B(p), ..., p \in A \cap B$



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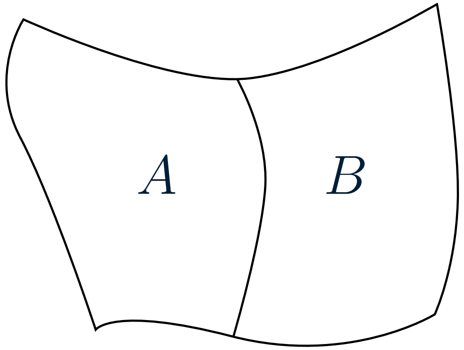
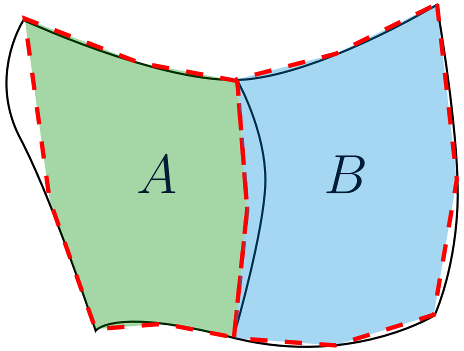


- conforming FEM
  - limited solution space $u(p) = \sum_i c_i N_i(p)$
  - full boundary conforming $u_A(p) = u_B(p), ..., p \in A \cap B$



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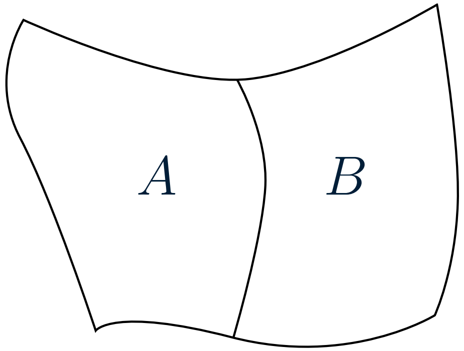

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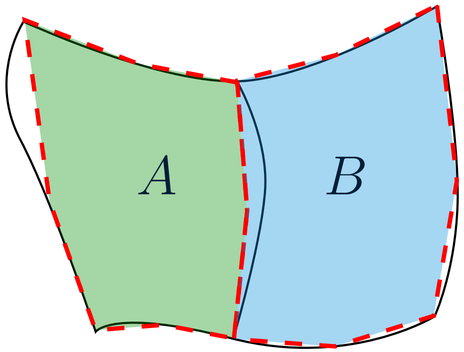
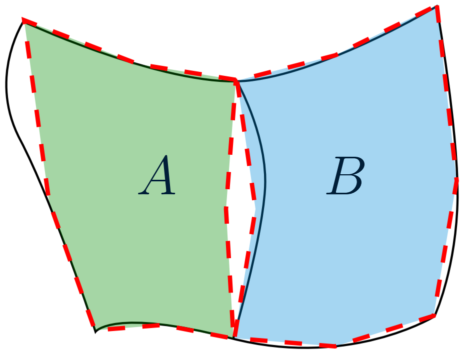


- non-conforming FEM
  - limited solution space $u(p) = \sum_i c_i N_i(p)$
  - partial boundary conforming $u_A(p) = u_B(p), p \in C \subset A \cap B$



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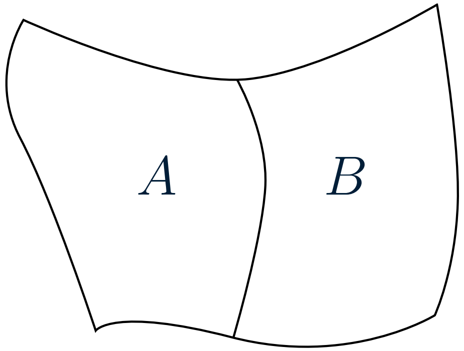
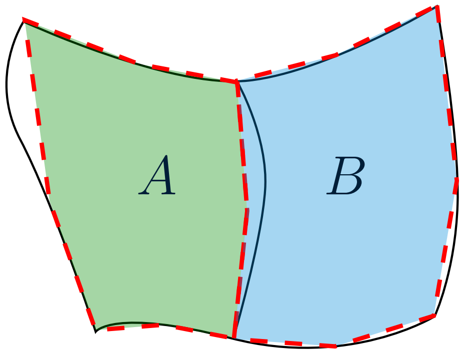


- conforming FEM
  - limited solution space
    $u(p) = \sum$ Stiffer $)$
  - full boundary conforming
    $u_A(p) = u_B(p), ..., p \in A \cap B$



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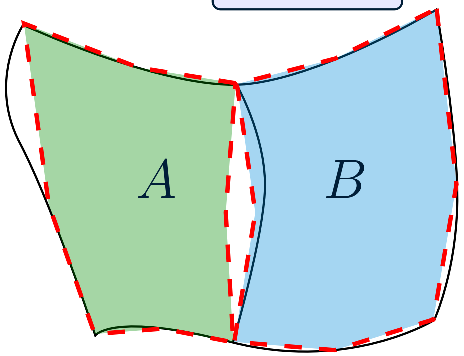

In Summary

- Intra-element: not enough DOFs in each element
  - stiffer in each element
- Inter-element: missing conformity constraints between elements
  - softer among elements
- Striking a balance by asymptotically approximating the continuous case

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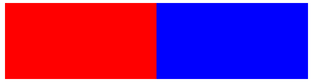

intra-element stiffness inter-element discontinuity

How To Make It Work?

- Intra-element stiffness
  - scalar shape functions $N(x)$ lacks enough DOFs, too stiff
  - use matrix-valued shape functions for more DOFs
  - with certain geometric conditions

scalar-valued

$$N(X) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}$$

matrix-valued

$$N(X) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

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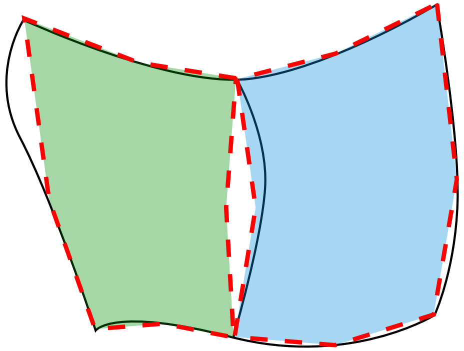
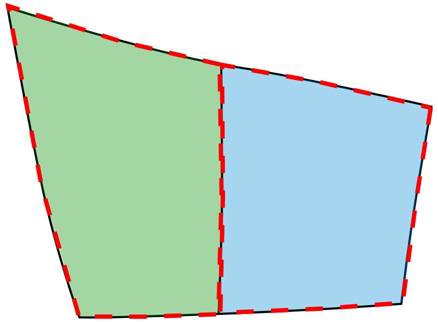
- Inter-element continuity
  - too much discontinuity will over-soften the system
  - partially conforming for representative deformations

non-conforming in general



conforming for representative cases



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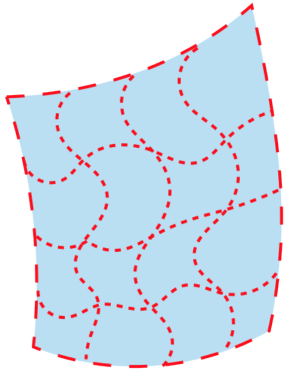
- Inter-element continuity
  - too much discontinuity will over-soften the system
  - partially conforming for representative deformations
- Deformation regularization for the remaining DOFs
  - small strain or small energy

Without deformation reg.



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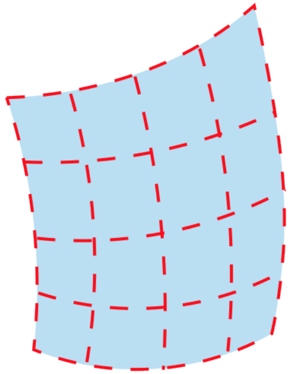


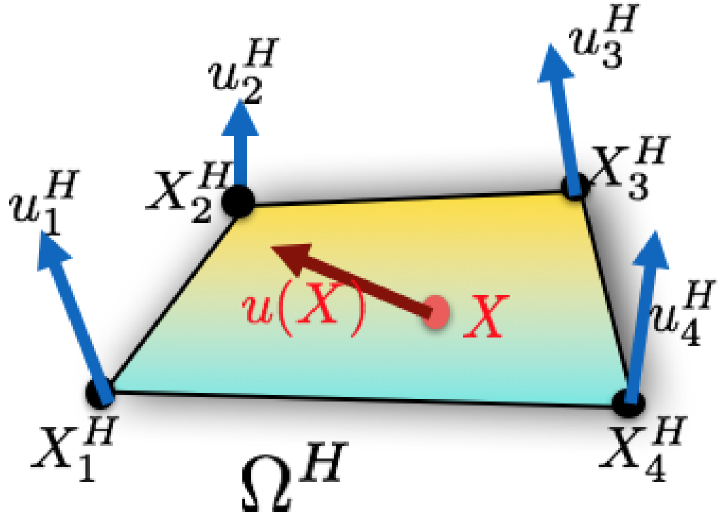
Matrix-valued Shape Function

- For every coarse element Ω^H
  - each vertex i is equipped with a shape function

    $$N_i^H : \Omega^H \to \mathbb{R}^{d\times d}$$

  - element-wise interpolation

    $$u(X) = \sum_{X_i \in \Omega^H} N_i^H(X) u_i^H, \quad \forall X \in \Omega^H$$



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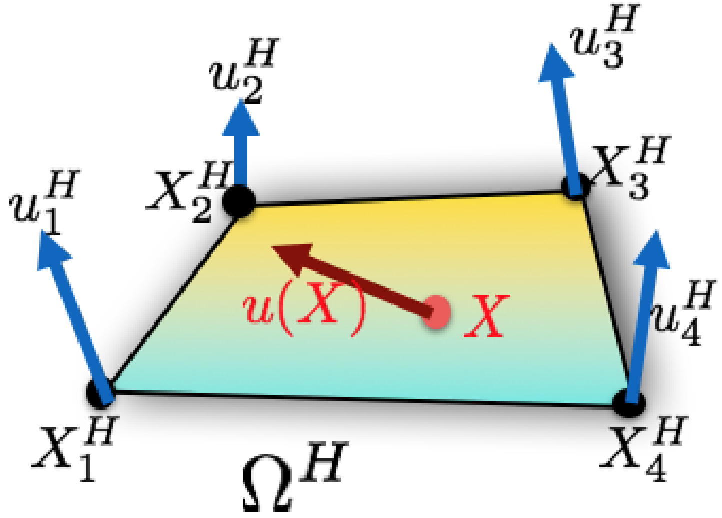
    general anisotropic shape functions N_i^H needs a local frame!



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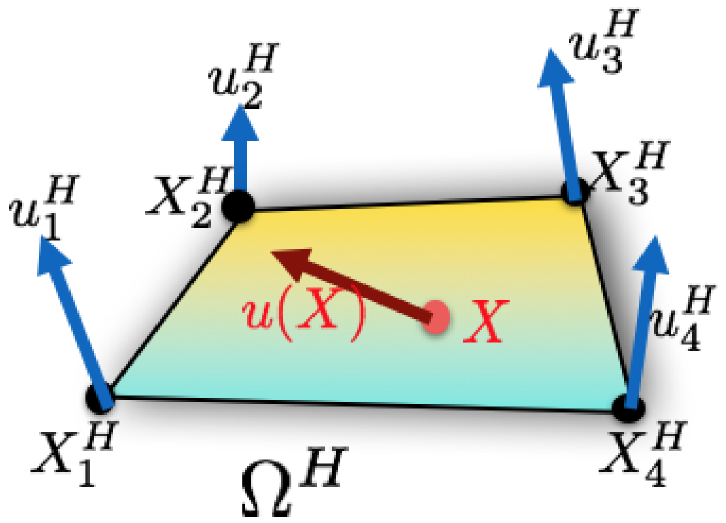
  - generalized corotational treatment

    $$u(X) = R_{\Omega^H}\left[X + \sum_i N_i^H(X)(R_{\Omega^H}^T x_i^H - X_i^H)\right] - X$$

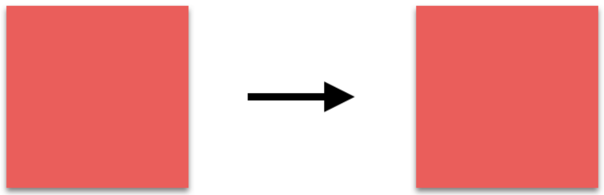
    - R_{Ω^H} comes from the polar decomposition of average deformation gradient



Adding Geometric Constraints

- Geometric conditions
  - translational invariance

$$\sum_i N_i^H(X) = \mathbb{I}$$



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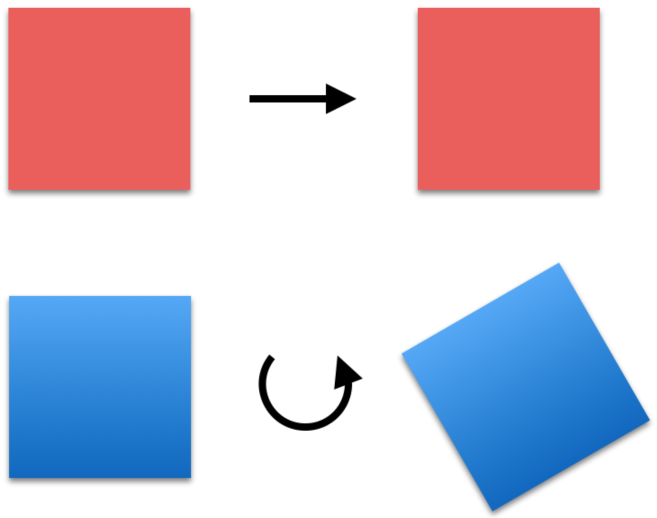
  - rotational invariance

$$\sum_i N_i^H(X)[X_i^H]_\times = [X]_\times$$



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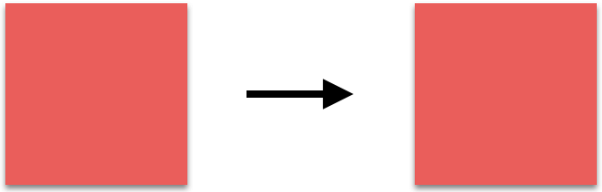

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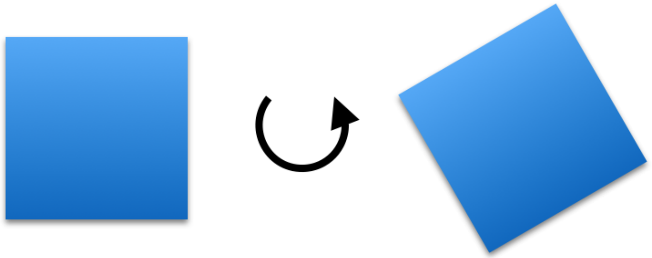
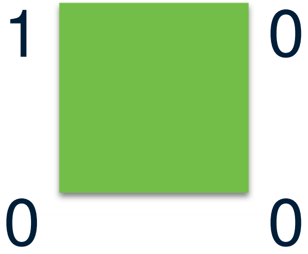


  - node interpolation

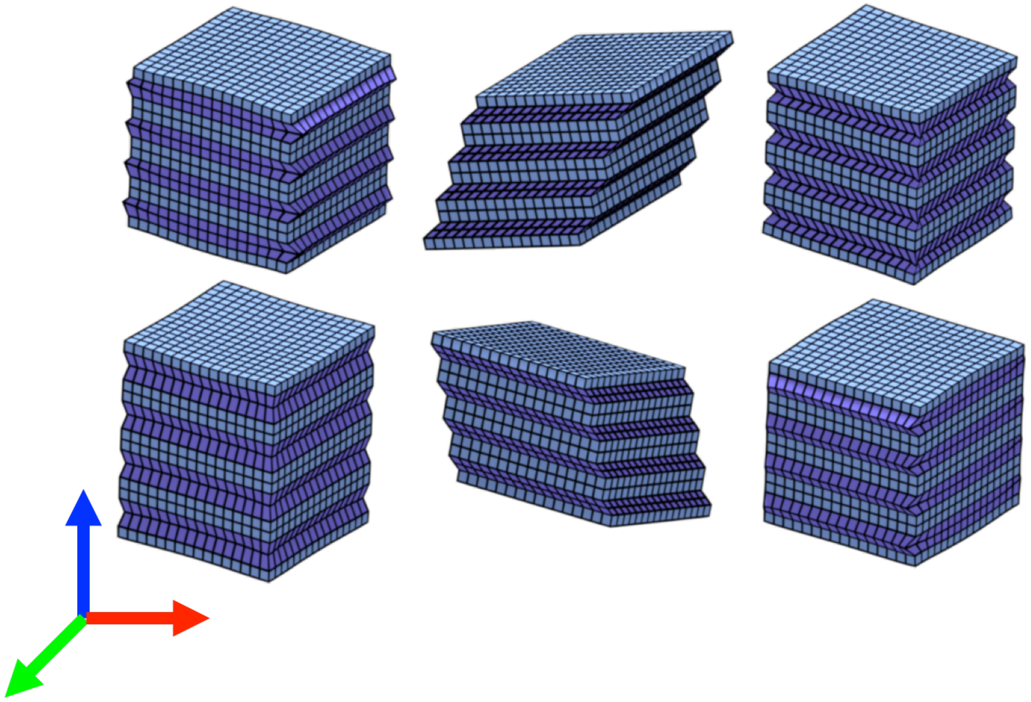
$$N_i^H(X_j^H) = \delta_{ij}\mathbb{I}$$



Partially Conforming Conditions

- Make sure basic deformations are perfectly reproduced
- Compute “representative” deformations
  - global harmonic displacements
  - using stiffness at rest shape
- Enforce exact reproduction
  - 6 more constraints for each element

$$h_{ab}(X) = \sum_i N_i^H(X) h_{ab}(X_i^H)$$



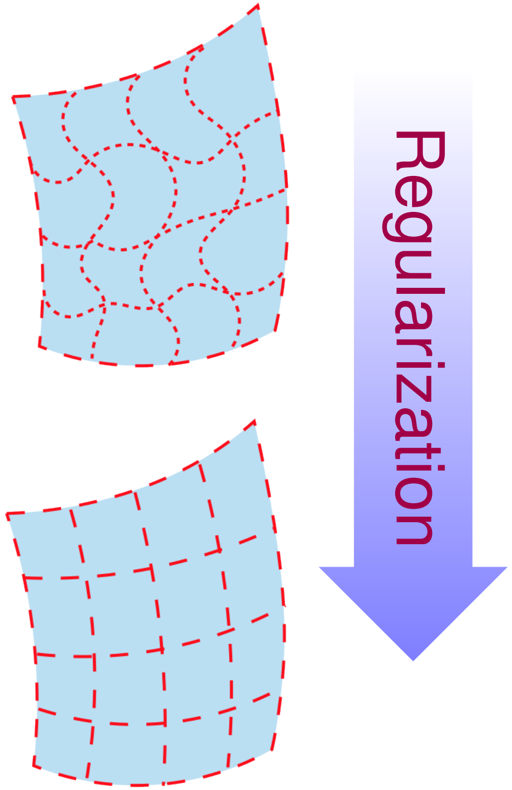
[Kharevych 2009]

Deformation Regularization

- For the remaining DOFs, favor the shape functions leading small deformation
- In the set of all possible solutions, take the nicest ones

$$\int_{\Omega} \mathrm{tr}\left((\nabla N_i^H)^T : \boxed{M} : \nabla N_i^H \right) dX$$

rank-4 tensor



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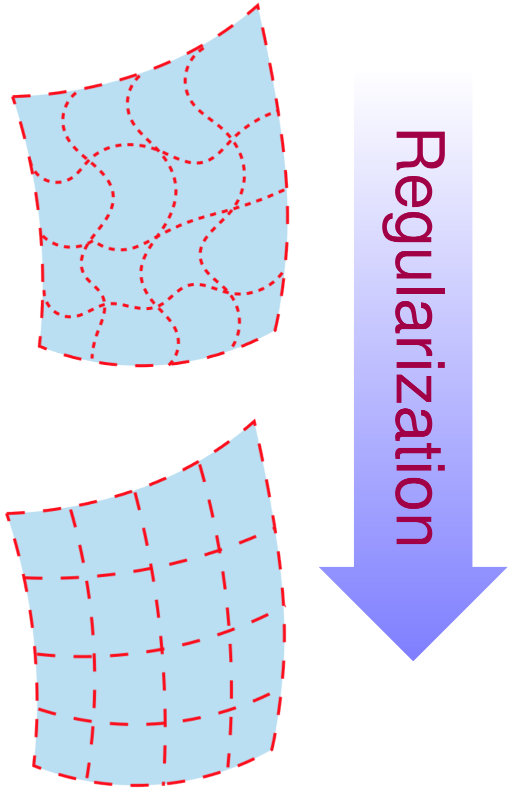
rank-4 tensor

- Two obvious options
  - harmonic conditioning $M = \mathbb{I}$: small strain
  - Ψ-harmonic conditioning $M = \partial^2\Psi/\partial(\nabla u)^2|_{u=0}$: small energy



Putting It All Together

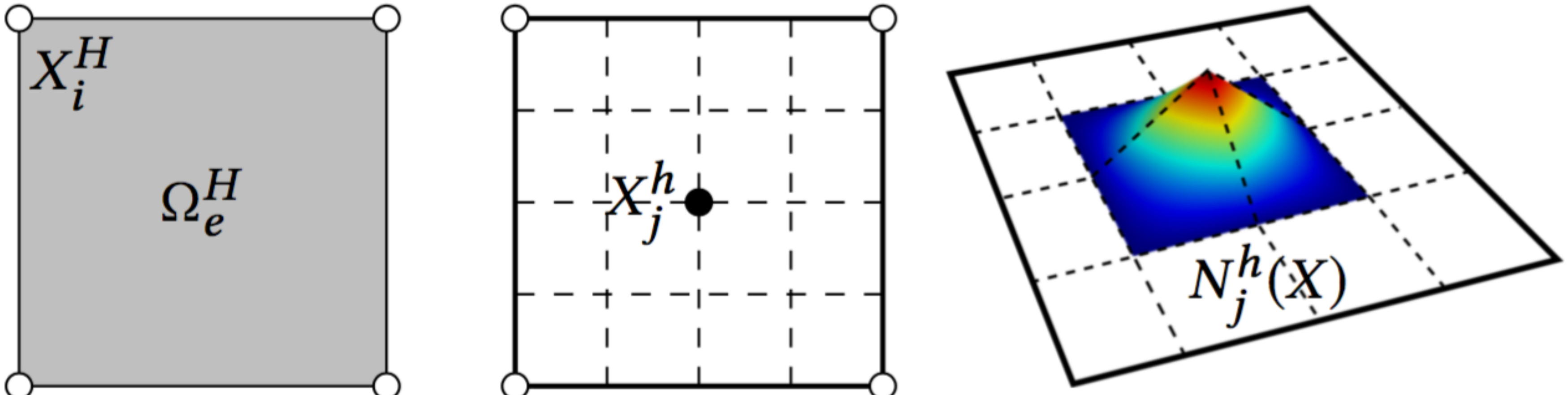
- Computing shape functions amount to an optimization

$$\min_{N} \int_{\Omega} \operatorname{tr}\left((\nabla N_i^H)^T : M : \nabla N_i^H\right) dX$$
$$\text{s. t.} \sum_i N_i^H(X) = \mathbb{I}$$
$$\sum_i N_i^H(X)[X_i^H]_\times = [X]_\times$$
$$N_i^H(X_j^H) = \delta_{ij}\mathbb{I}$$
$$\sum_i N_i^H(X) h_{ab}(X_i^H) = h_{ab}(X)$$

- constrained quadratic programming per element
- can proceed in parallel

Basis Discretization

- Each component of our matrix-valued basis functions is discretely represented using the fine mesh basis functions



piecewise bilinear function

$$N_i^H(X)_{p,q} = \sum_j n_{ij,pq} N_j^h(X)$$

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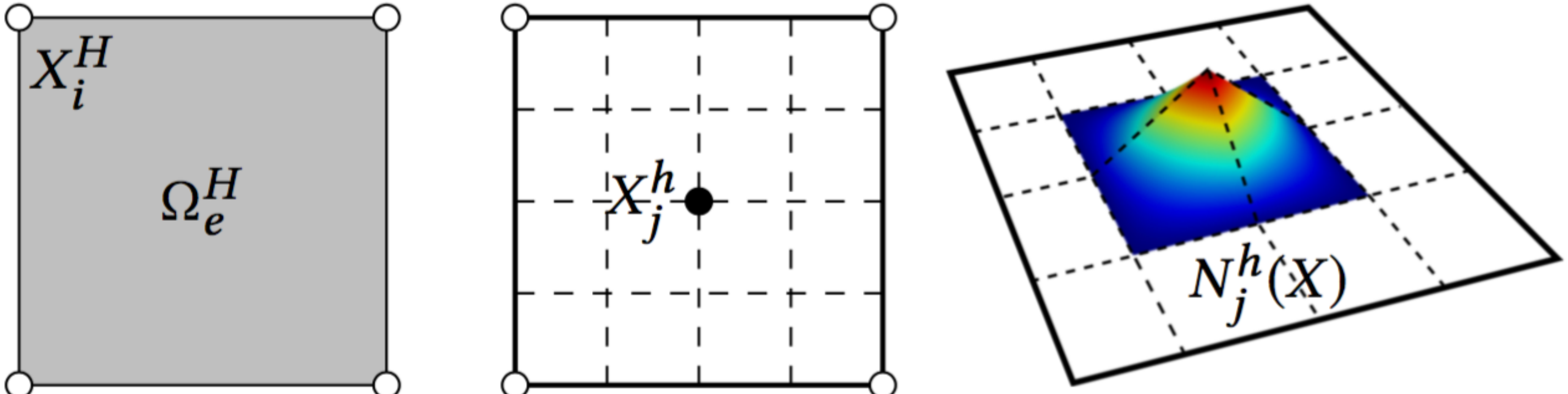

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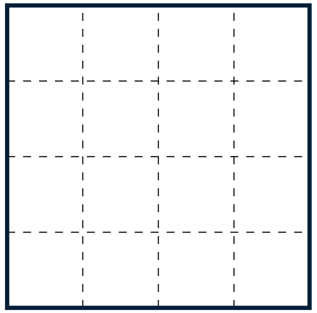
DOFs for basis optimization

Remarks on DOFs

- Note that the DOFs for basis optimization and DOFs for coarse simulation are different.
- E.g. for coarse mesh with a single element (2D)
  - basis optimization
    - DOFs as matrix shape functions: $n_{ij,pq}$



$$\underset{i}{4} \times \underset{j}{25} \times \underset{p}{2} \times \underset{q}{2} = 400$$

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- E.g. for coarse mesh with a single element (2D)
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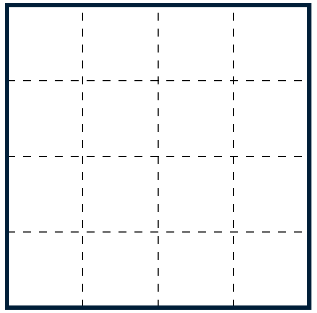
  - coarse simulation
    - DOFs as displacements: $u_{i,j}$



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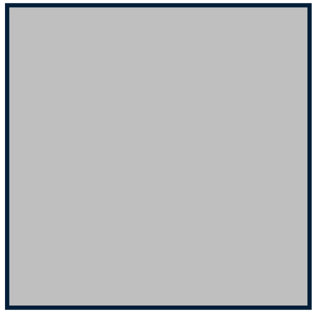

$$\underset{i}{4} \times \underset{j}{2} = 8$$

The Balance

- The coarse conforming element are generally stiffer in each element

intra-element stiffness — inter-element discontinuity

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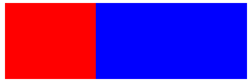

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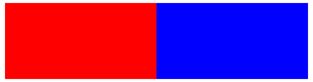
The Balance

- The coarse conforming element are generally stiffer in each element
  - we soften it via matrix-valued shape functions
- Discontinuous elements are usually too soft among elements
  - we impose partial conforming constraints
- Fine tune the balance
  - we regularize the deformation (small strain or energy) using the remaining DOFs



intra-element stiffness   inter-element discontinuity

Illustration of Local/Global Spectrum

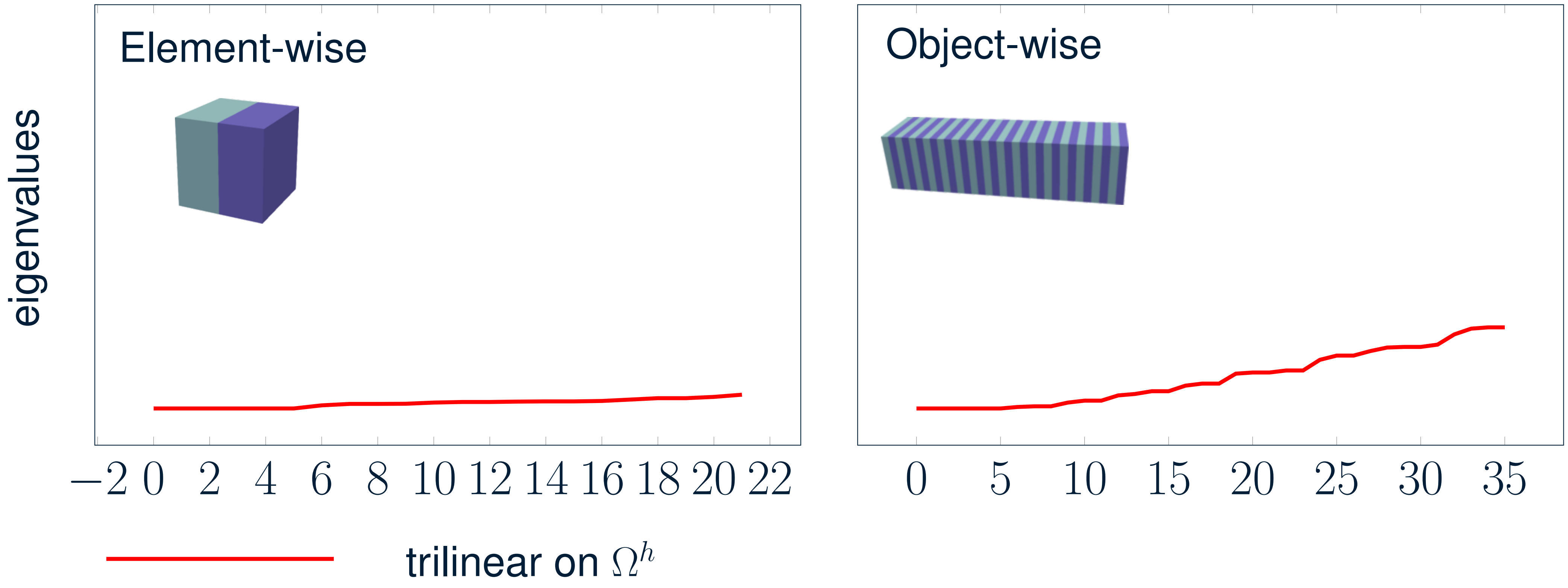


Illustration of Local/Global Spectrum

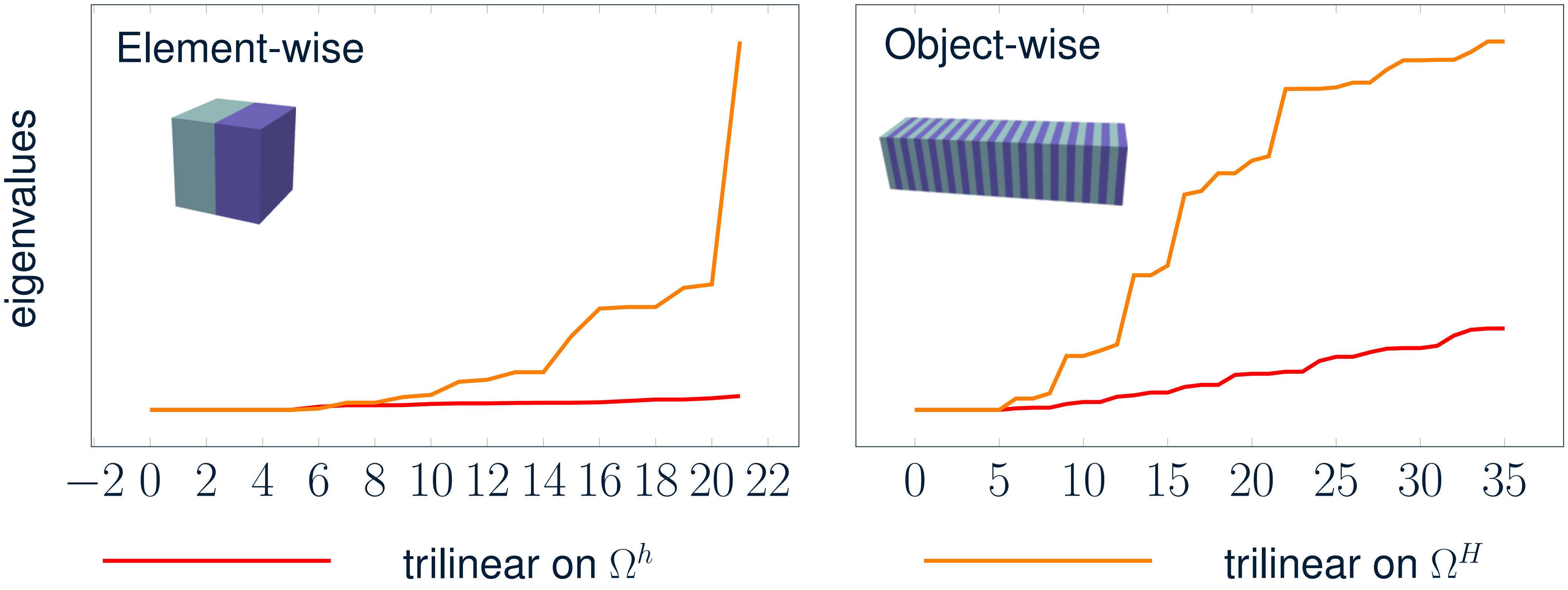


Illustration of Local/Global Spectrum

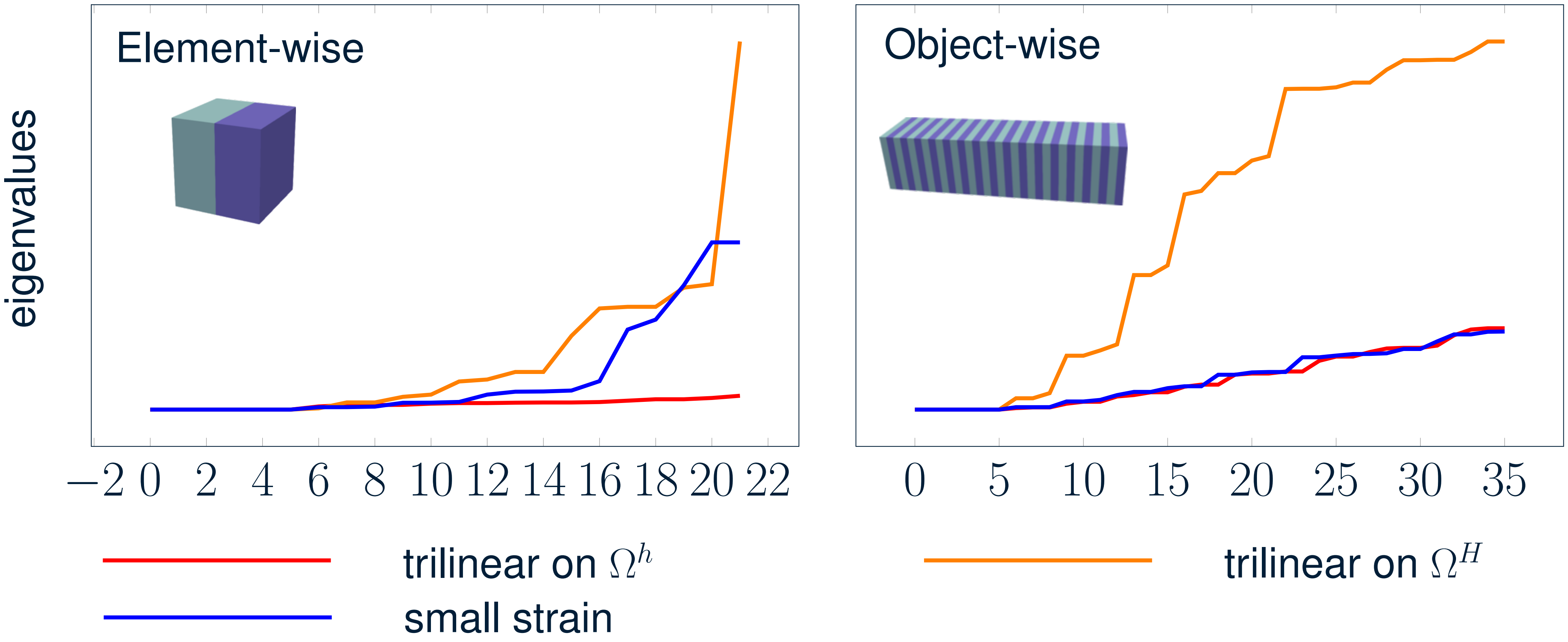


Illustration of Local/Global Spectrum

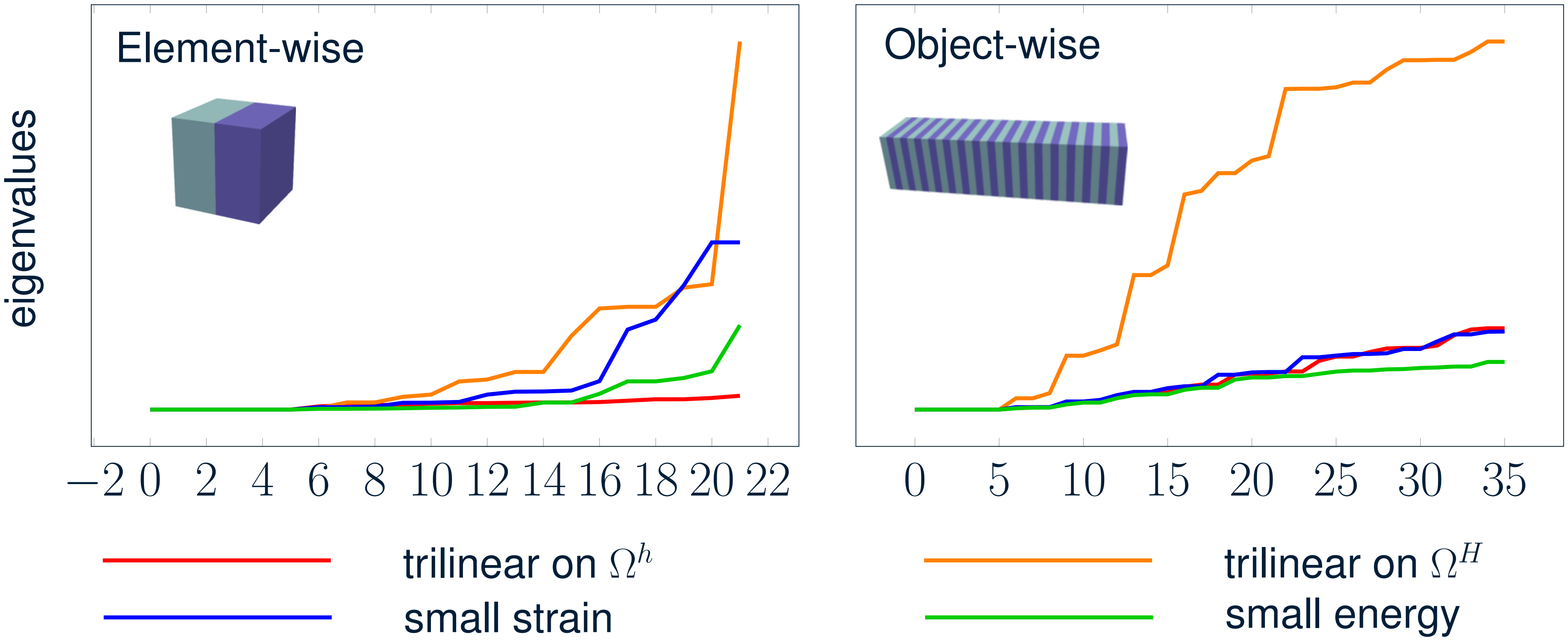
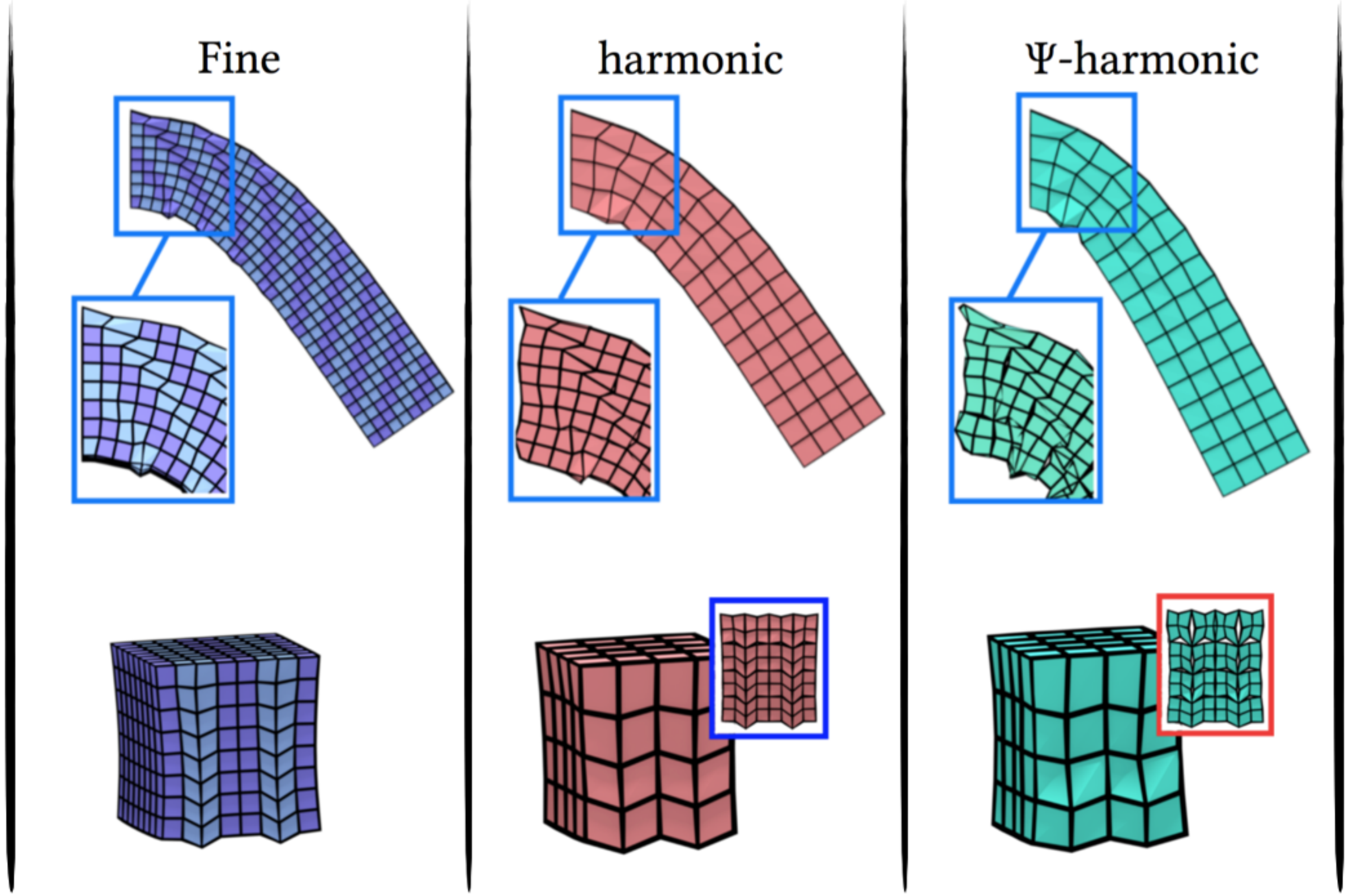


Illustration of Resulting Deformation



Simulation

- Calculation of deformation gradient

$$\nabla_X x = \nabla_X u + \mathbb{I} = (R_e - \mathbb{I}) + \sum_i R_e \otimes (R_e^T x_i - X_i) : \frac{\partial N_i^H}{\partial X} + \mathbb{I}$$

$$= R_e + \left(\sum_i R_e \otimes (R_e^T x_i - X_i) : \frac{\partial N_i^H}{\partial \xi} \right) \left(\sum_j X_j \frac{\partial \overline{N}_j^H}{\partial \xi} \right)^{-1}$$

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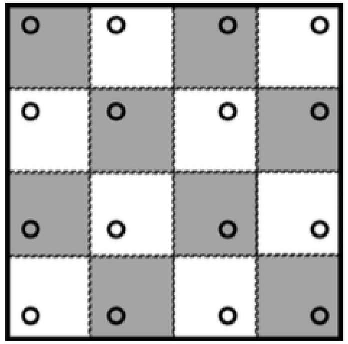
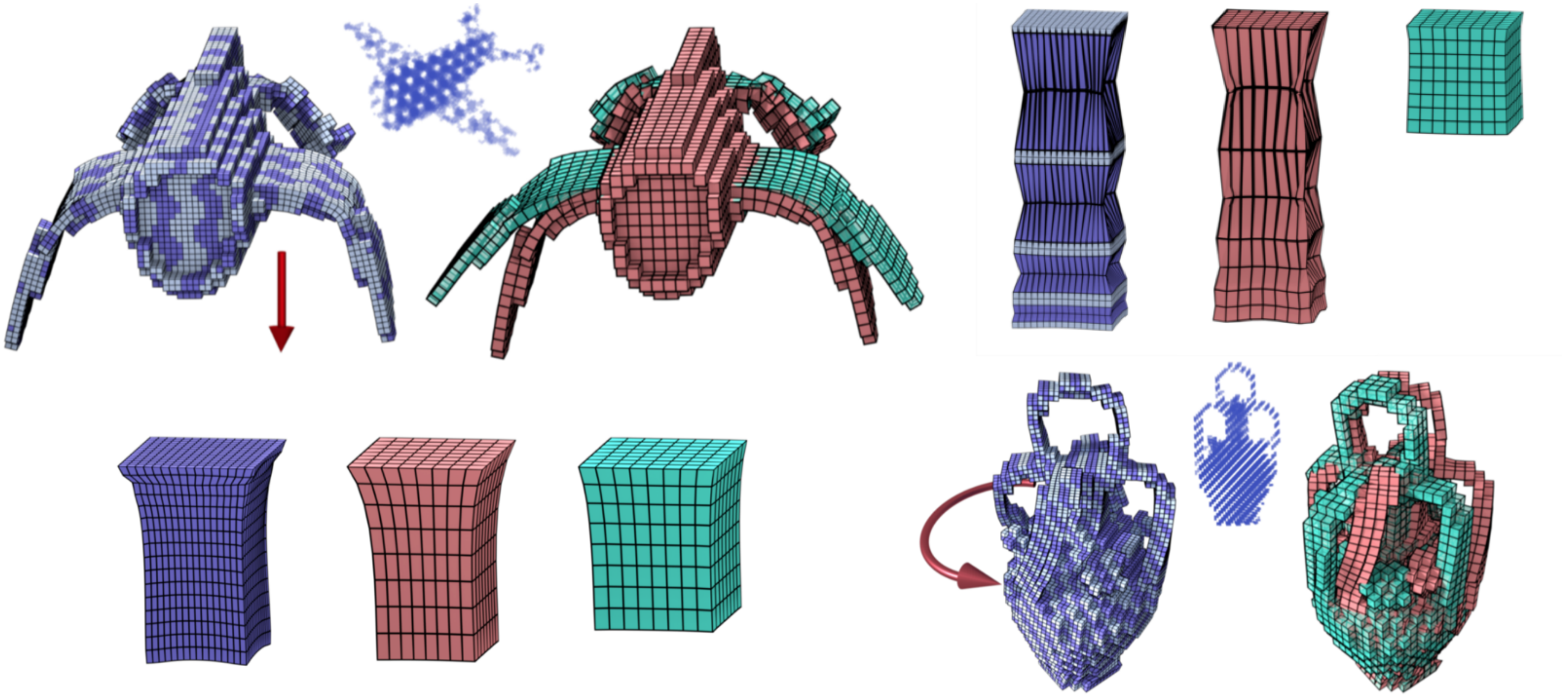
- Calculation of energy integral
  - standard Gaussian-Legendre quadrature

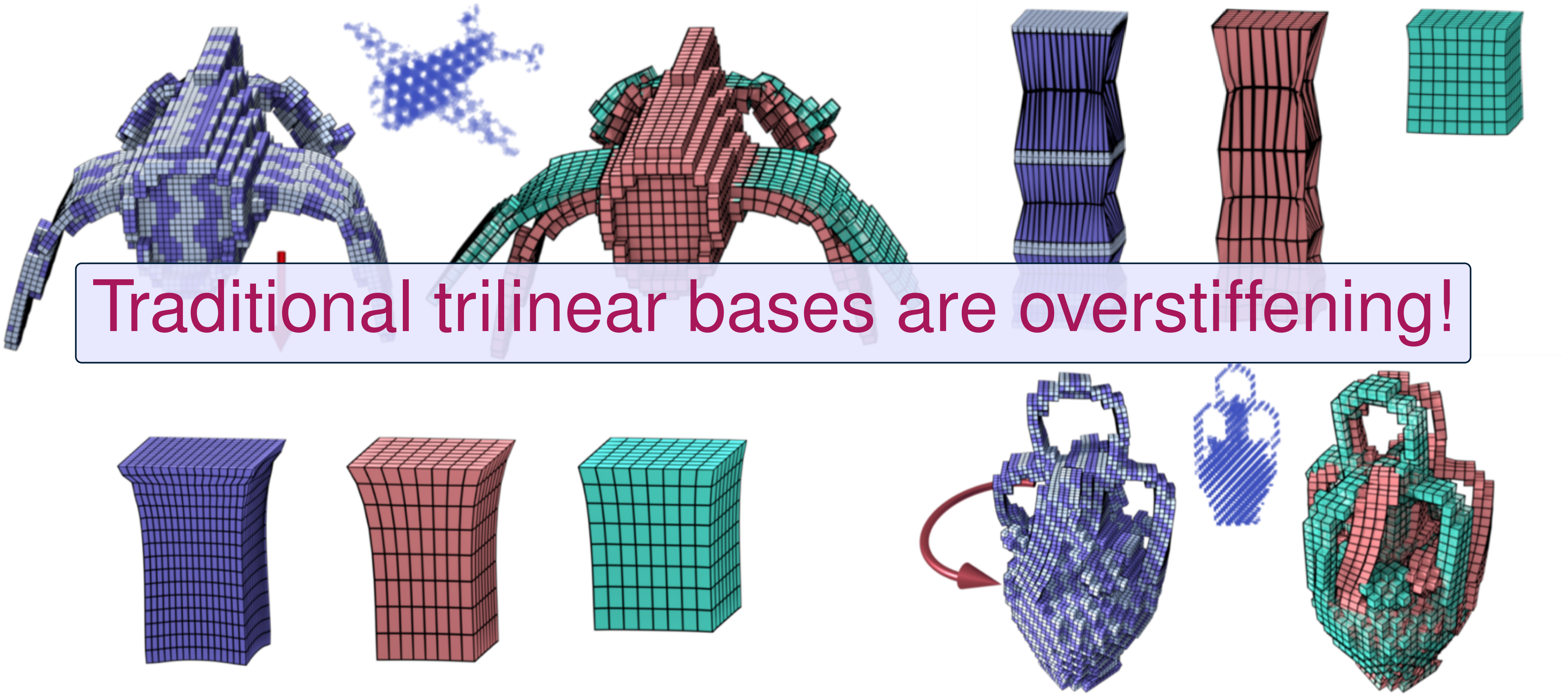


Results

Comparison with Trilinear basis

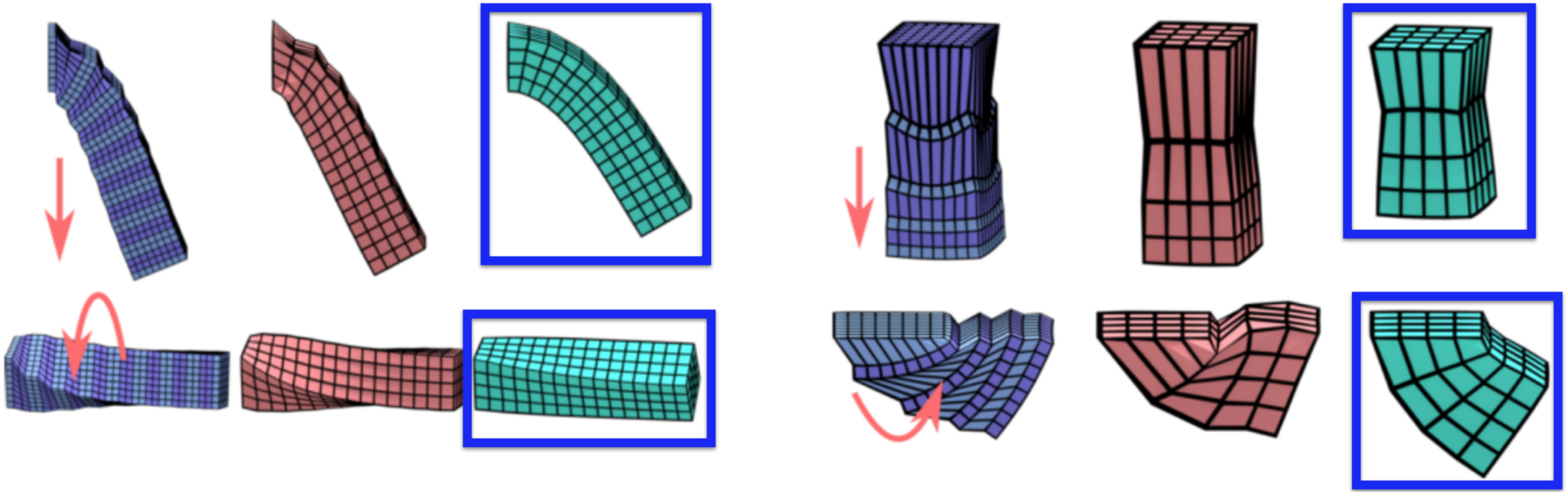


Comparison with Trilinear basis



Traditional trilinear bases are overstiffening!

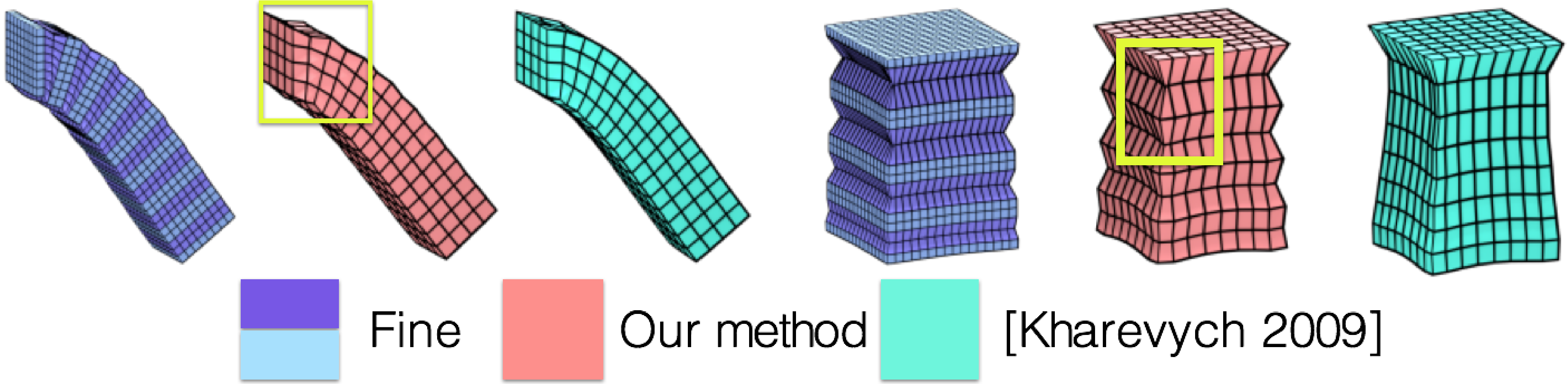
Comparison with [Nesme 2009]



[Nesme 2009] uses diagonal shape function and strong conforming conditions.

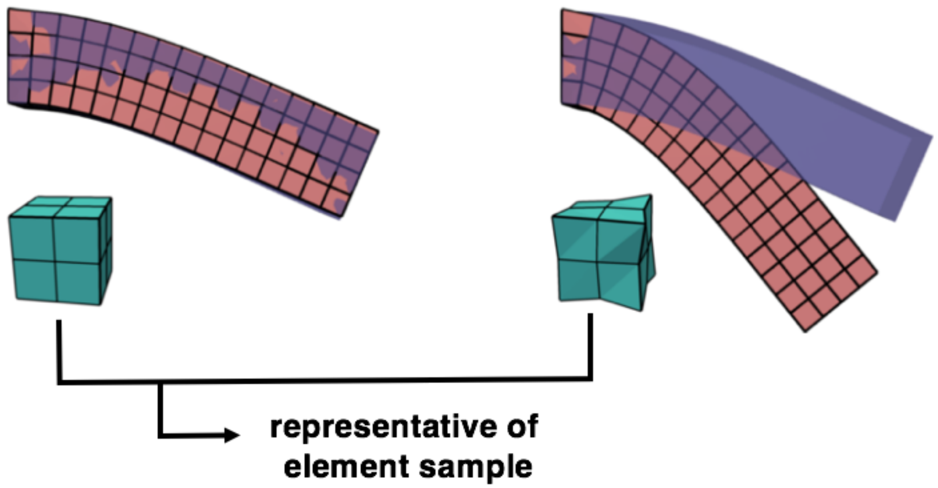
[Nesme 2009] is too stiff as well

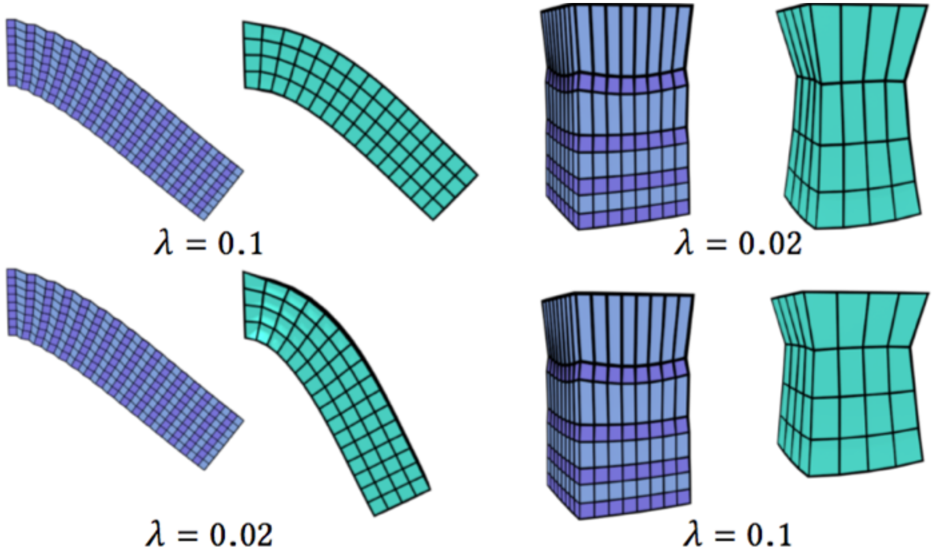
Comparison with [Kharevych 2009]



We capture the details better even on linear elasticity

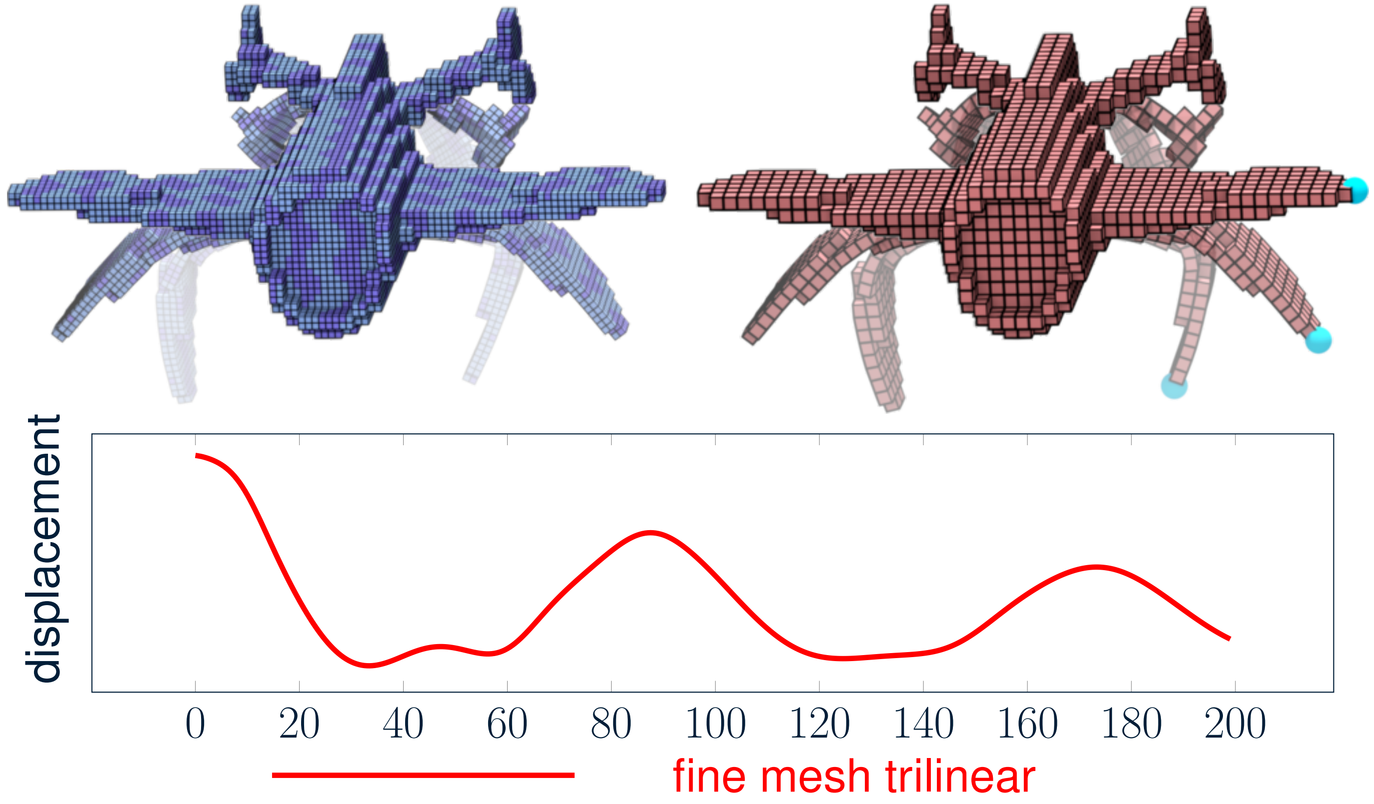
Comparison with [DDFEM]



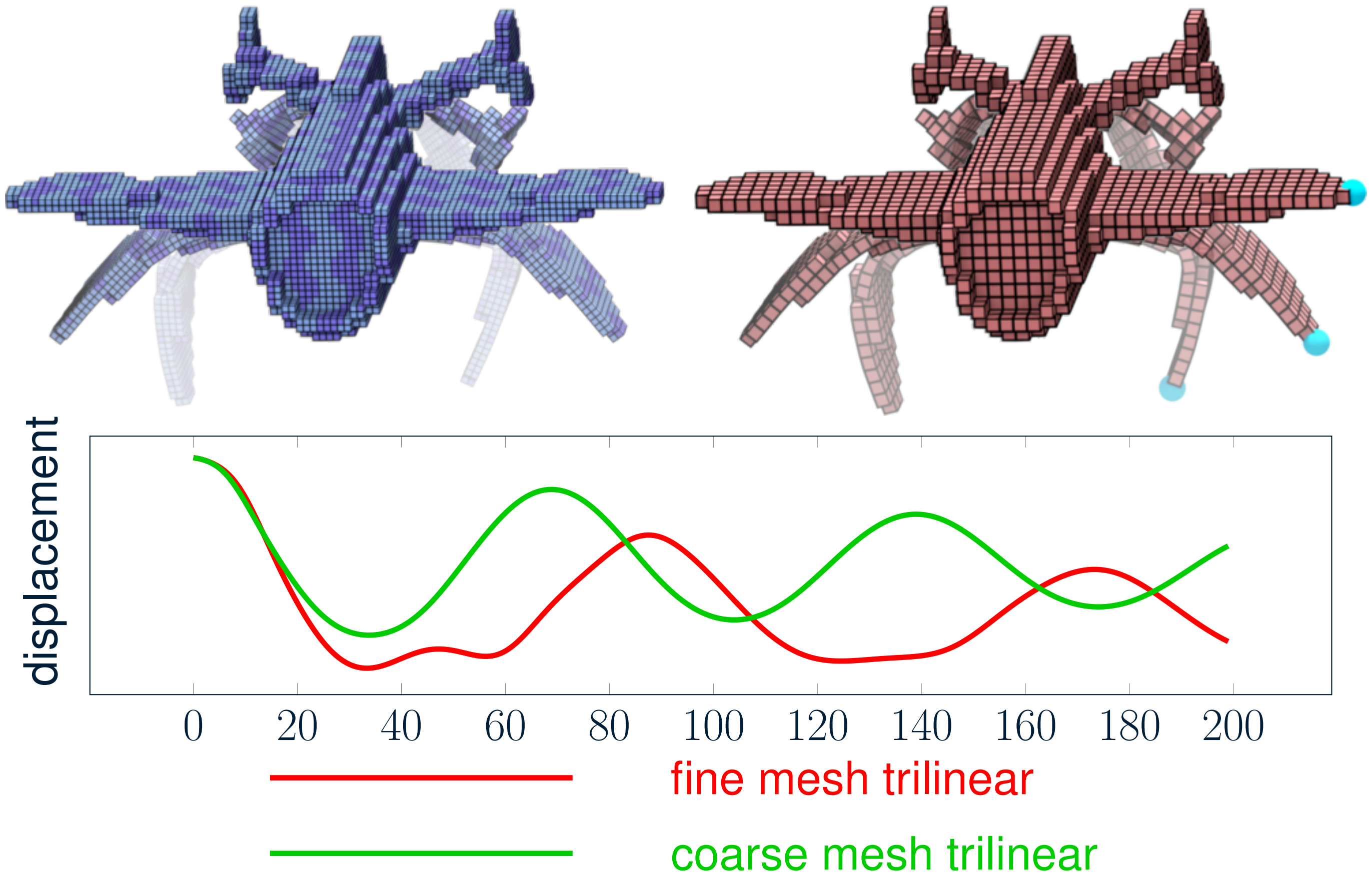


[DDFEM] relies on dataset and parameter tuning

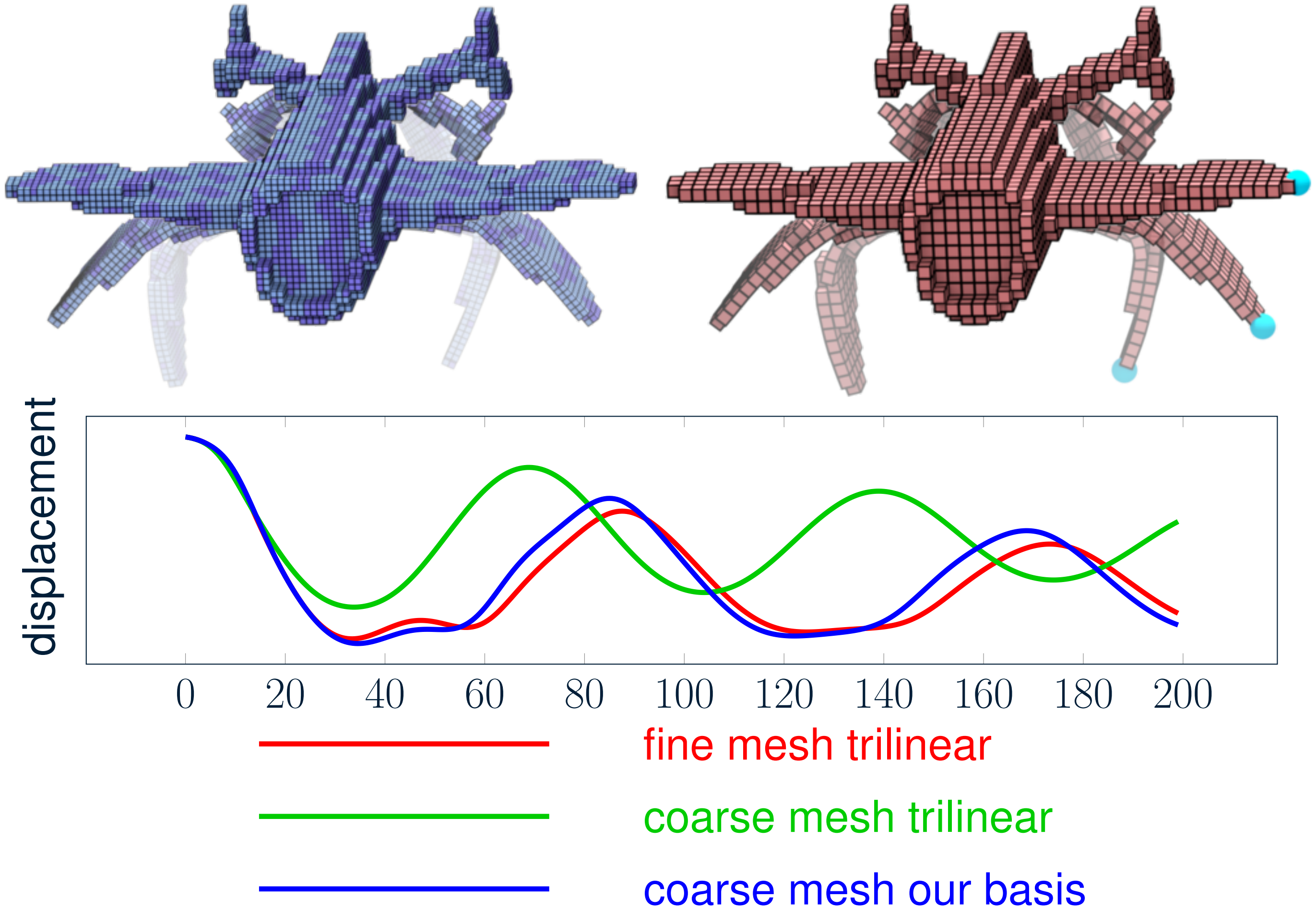
Dynamics


displacement
0
20
40
60
80
100
120
140
160
180
200
fine mesh trilinear

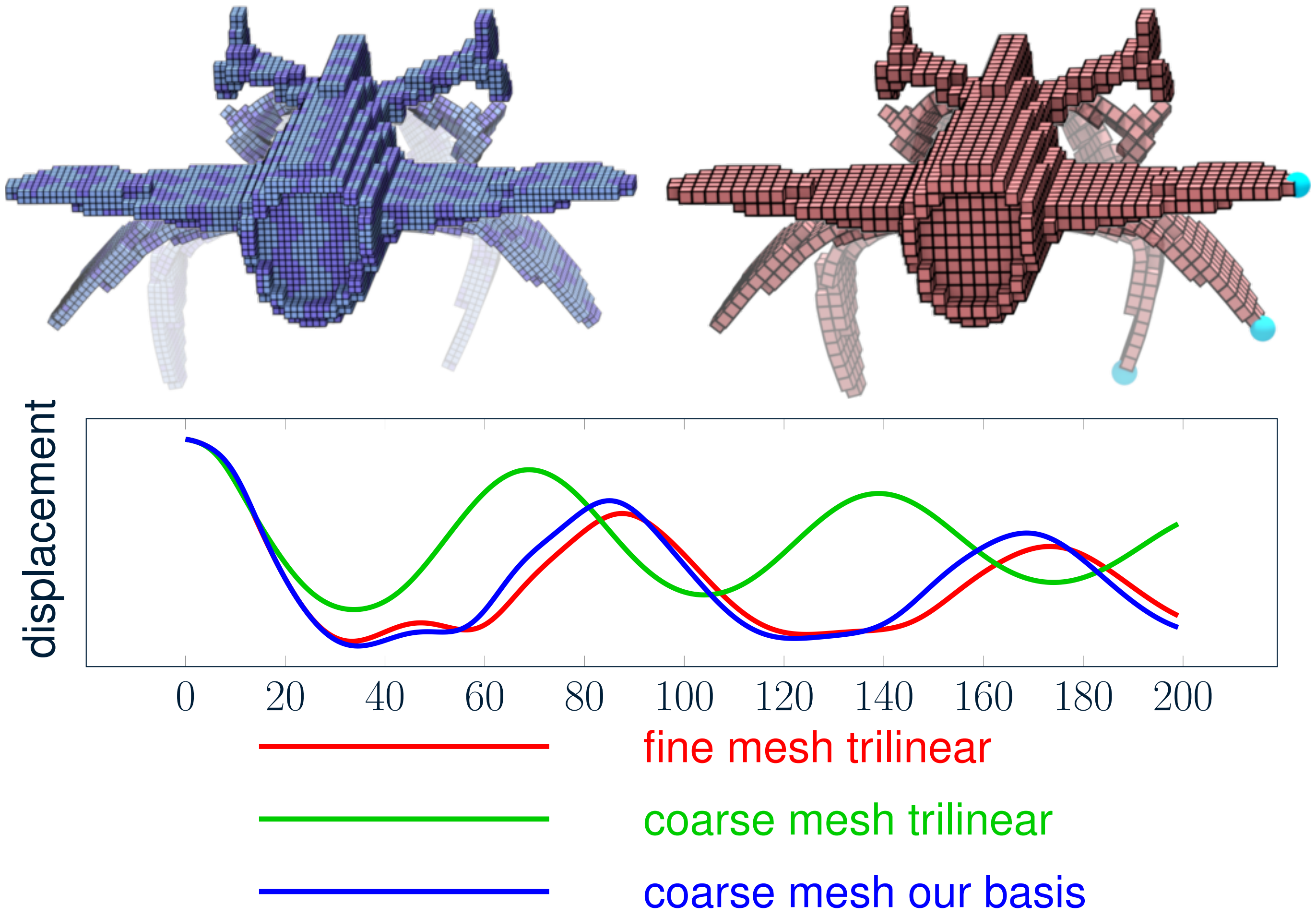
Dynamics


displacement
0
20
40
60
80
100
120
140
160
180
200
fine mesh trilinear
coarse mesh trilinear

Dynamics


displacement
0
20
40
60
80
100
120
140
160
180
200
fine mesh trilinear
coarse mesh trilinear
coarse mesh our basis

Dynamics


displacement
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fine mesh trilinear
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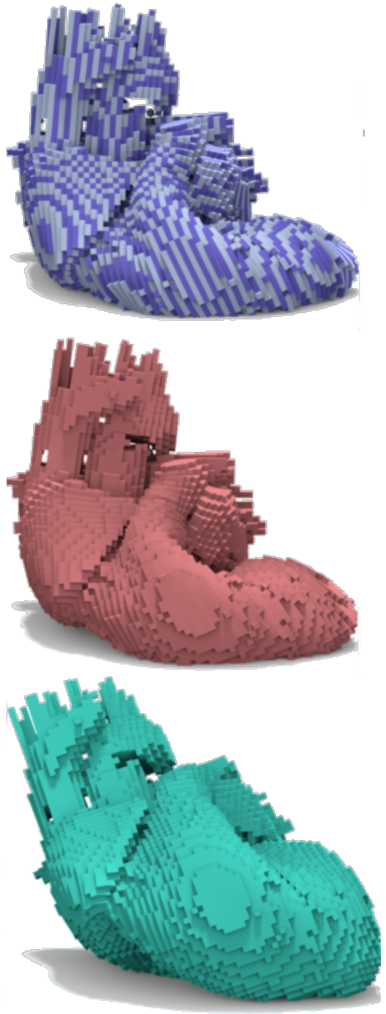
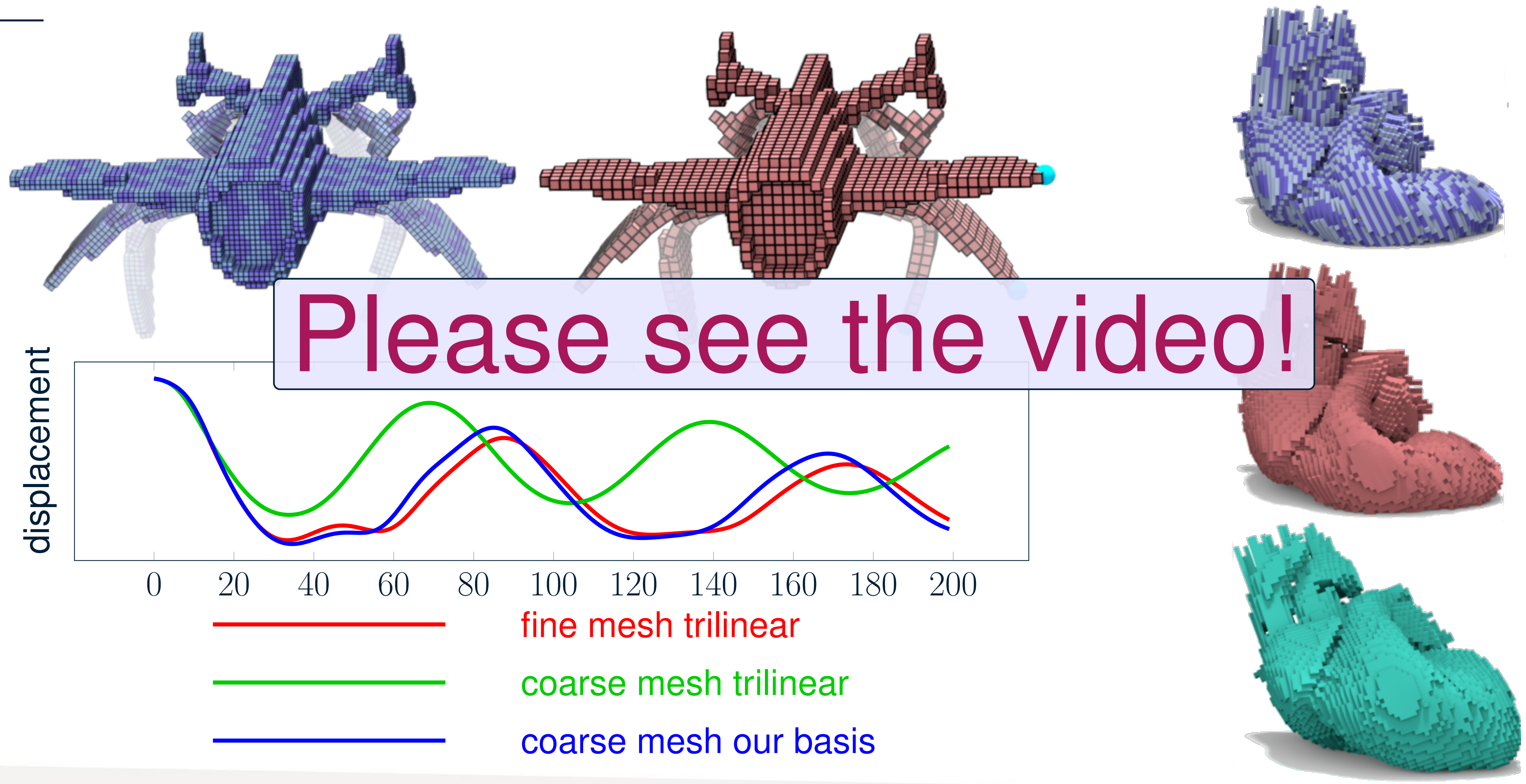
coarse mesh our basis

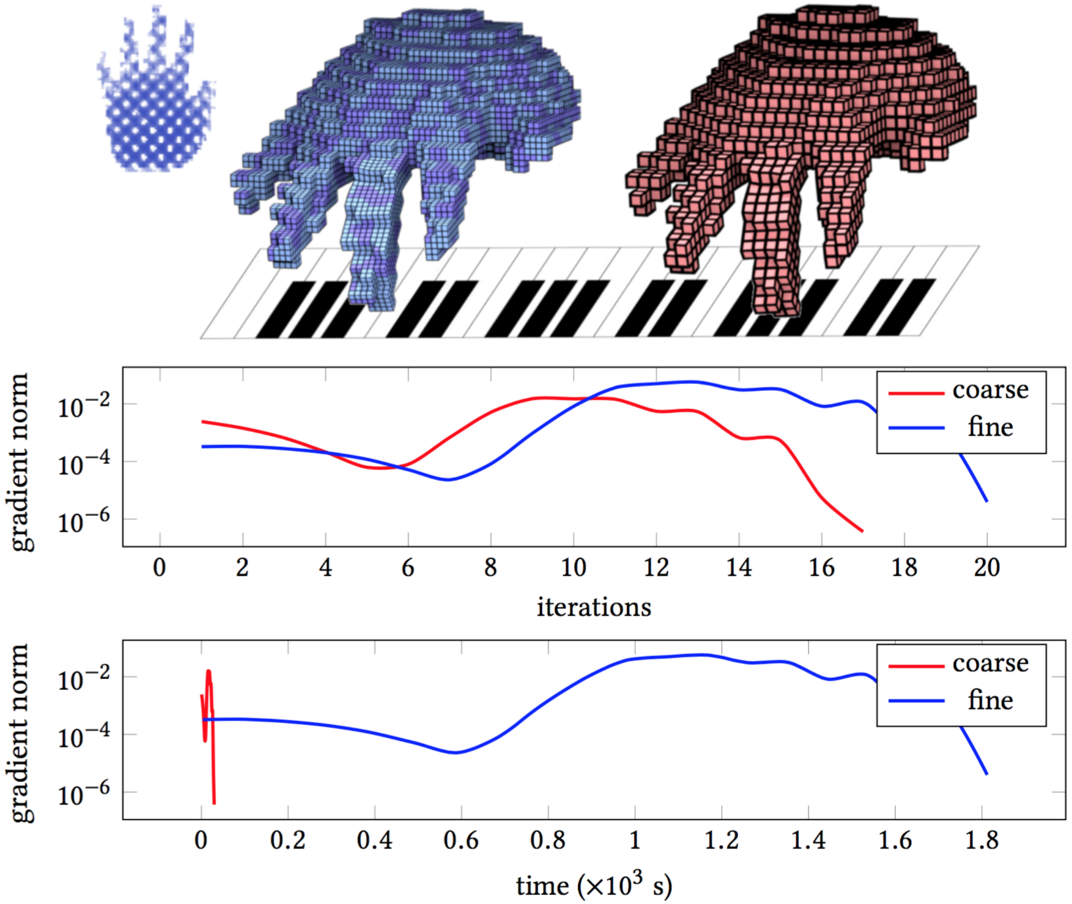


Dynamics



Please see the video!
displacement
0
20
40
60
80
100
120
140
160
180
200
fine mesh trilinear
coarse mesh trilinear
coarse mesh our basis

Performance



Fine mesh: 31337 verts, 26176 cells

Coarse mesh: 4627 verts, 3272 cells

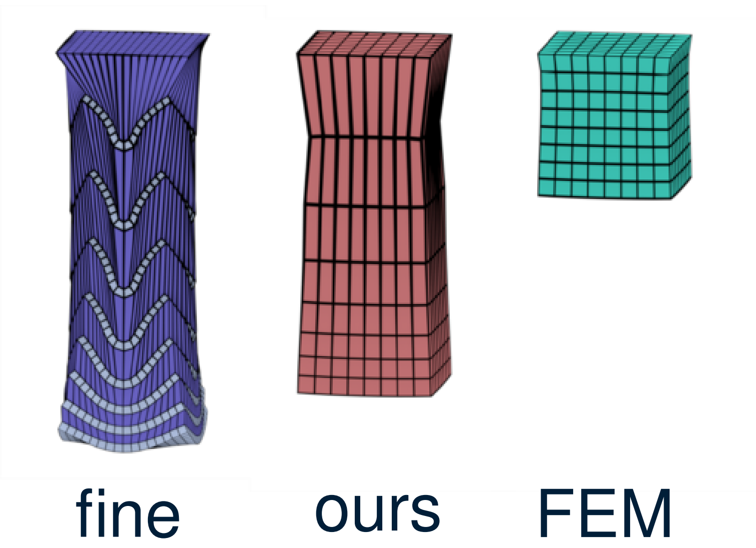
one level of coarsening

about 1/8 nodes

60x times faster!

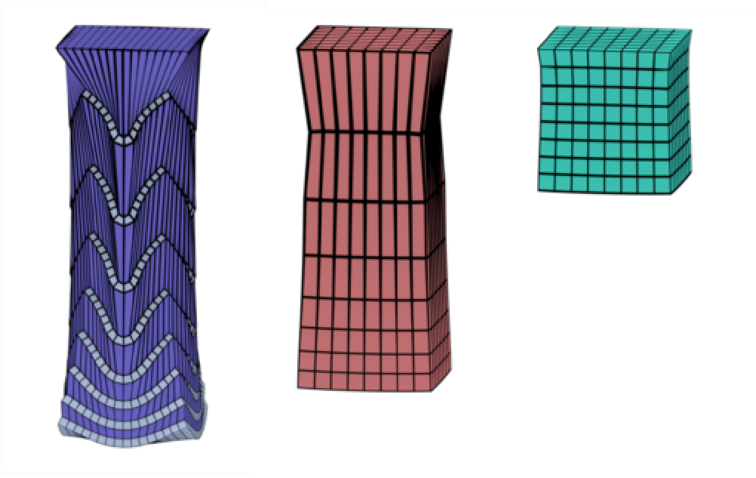
Limitations and Future Work

- Function of shape functions for very large deformation



Limitations and Future Work

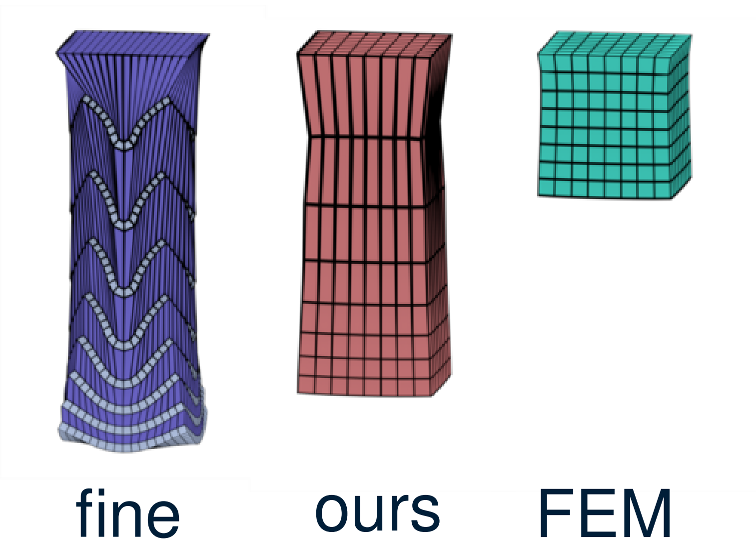
- Function of shape functions for very large deformation
- Coarsening of dynamical system with inhomogeneous mass distribution



fine ours FEM

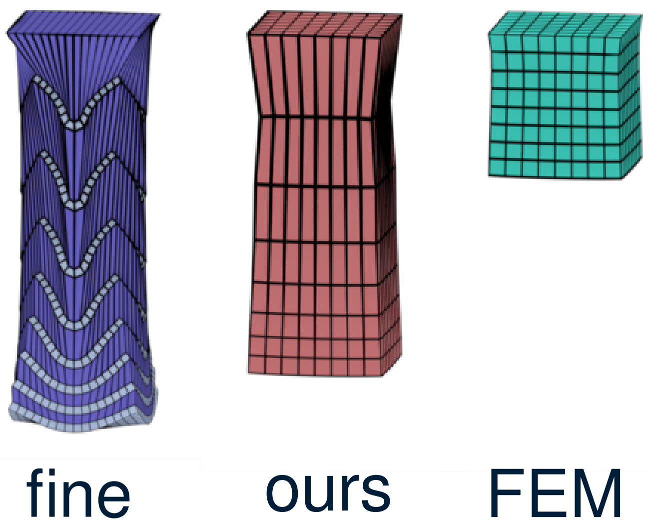
Limitations and Future Work

- Function of shape functions for very large deformation
- Coarsening of dynamical system with inhomogeneous mass distribution
- Coarsening of boundary conditions



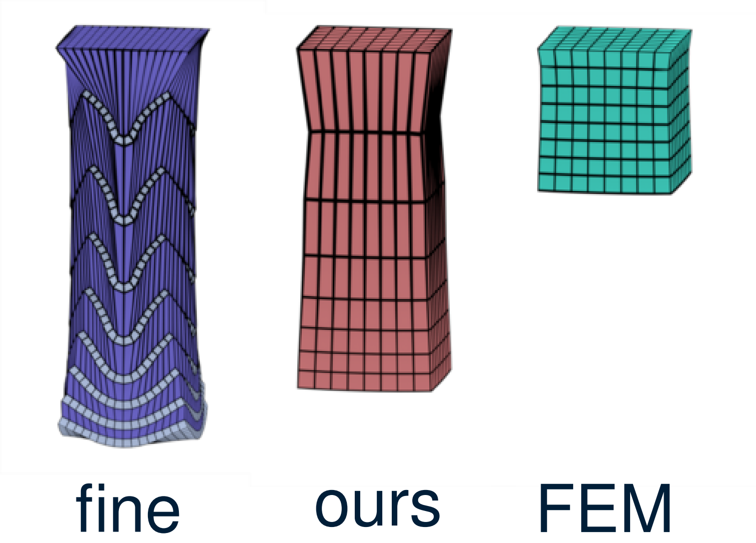
Limitations and Future Work

- Function of shape functions for very large deformation
- Coarsening of dynamical system with inhomogeneous mass distribution
- Coarsening of boundary conditions
- Better cubature schemes



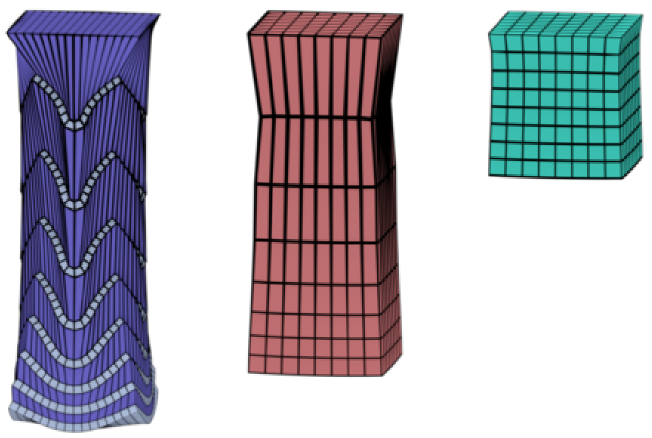
Limitations and Future Work

- Function of shape functions for very large deformation
- Coarsening of dynamical system with inhomogeneous mass distribution
- Coarsening of boundary conditions
- Better cubature schemes
- Space time coarsening



Limitations and Future Work

- Function of shape functions for very large deformation
- Coarsening of dynamical system with inhomogeneous mass distribution
- Coarsening of boundary conditions
- Better cubature schemes
- Space time coarsening
- ...



Thank You!

Q&A